DIFFERENTIAL EQUATION

Introductory Topics

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OBJECTIVES

At the end of the lesson, students must be able to:

1. Determine the type, order, degree and linearity of a differential equation by identification of each features in a given differential equation correctly.

INTRODUCTION

Introduction

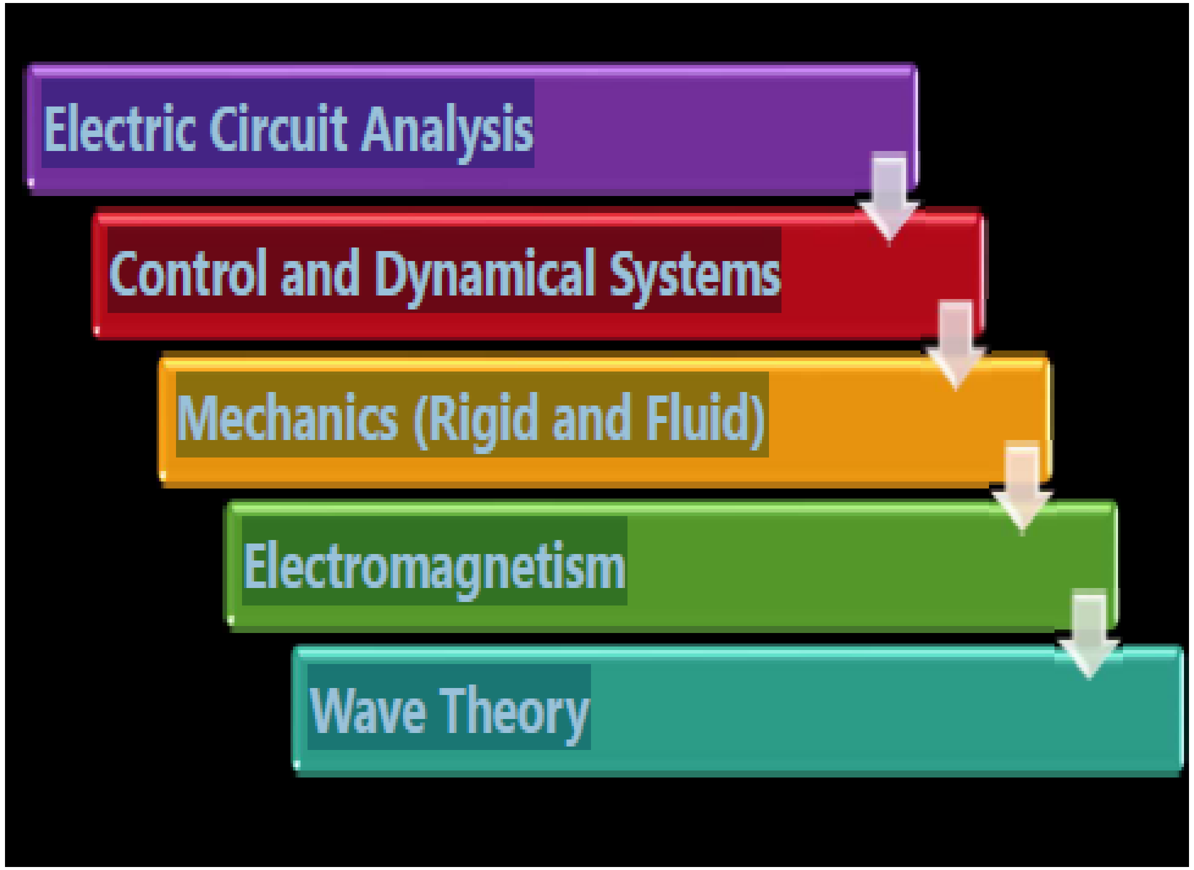
- Scientists and engineers develop mathematical models for physical processes as an aid to understanding and predicting the behavior of the processes.

- we discuss mathematical models that help us understand, among other things, decay of radioactive substances, electrical networks, population dynamics, dispersion of pollutants, and trajectories of moving objects.

- Modeling a physical process often leads to equations that involve not only the physical quantity of interest but also some of its derivatives.

INTRODUCTION

Differential equations is a powerful tool in solving many real-world problems in engineering, physical sciences,natural sciences, business and economics.



INTRODUCTION

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.

2. To find—either exactly or approximately—the appropriate solution of that equation.

3. To interpret the solution that is found.

INTRODUCTION

An equation relating an unknown function and one or more of its derivatives is called a differential equation.

Other definition

- an equation containing *derivatives* or *differentials*
- expresses a relation between an *unknown function* and its *derivatives*.
- may contain one or more derivatives
- one or more independent variables

INTRODUCTION

Notation for Derivatives

Leibniz notation $\frac{d()}{d()}$,

$\frac{dy}{dx}$

$\frac{df}{dx}(x)$ or $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$.

Higher derivatives are written as

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}.$$

Lagrange's notation (prime marks),

y'

$f^{(4)}(x), f^{(5)}(x), f^{(6)}(x), \ldots.$

$f'(x)$.

$f^{iv}(x), f^{v}(x), f^{vi}(x), \ldots,$

INTRODUCTION

Euler's notation (D operator)

$$(Df)(x) = \frac{df(x)}{dx}.$$

Dy D^2y

Higher derivatives are notated as "powers" of *D* (where the superscripts denote iterated composition of *D*),

$D^2 f$ for the second derivative,

$D^3 f$ for the third derivative, and

$D^n f$ for the *n*th derivative.

Newton's notation

- also called the dot notation, fluxions, or sometimes, crudely, the flyspeck notation

$\dot{x}$ $\ddot{x}$

The first and second derivatives of *x*, Newton's notation.

if *y* is a function of *t*, then $\dot{y}$

Higher derivatives are represented using multiple dots, as in

$\ddot{y}, \dddot{y}$

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Partial derivatives notation

$$\frac{\partial f}{\partial x}$$

A function *f* differentiated against *x*.

$$\frac{\partial^2 u}{\partial x^2} = u_{xx}$$

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Examples of Differential Equation

$$\frac{dy}{dx} = \cos x, \quad (1)$$

$$\frac{d^2y}{dx^2} + k^2 y = 0, \quad (2)$$

$$(x^2 + y^2)\,dx - 2xy\,dy = 0, \quad (3)$$

$$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (4)$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = E\omega \cos \omega t, \quad (5)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad (6)$$

$$\left(\frac{d^2 w}{dx^2} \right)^3 - xy\frac{dw}{dx} + w = 0, \quad (7)$$

INTRODUCTION

Examples of Differential Equation

$$\frac{d^3x}{dy^3} + x\frac{dx}{dy} - 4xy = 0, \qquad (8)$$

$$\frac{d^2y}{dx^2} + 7\left(\frac{dy}{dx}\right)^3 - 8y = 0, \qquad (9)$$

$$\frac{d^2y}{dt^2} + \frac{d^2x}{dt^2} = x, \qquad (10)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \qquad (11)$$

INTRODUCTION

- When an equation involves one or more derivatives with respect to a particular variable, that variable is called an **independent variable**. A variable is called **dependent** if a derivative of that variable occurs.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\cos\omega t \qquad (5)$$

i is the dependent variable, t the independent variable, and L, R, C, E, and ω are called parameters.

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In the equation

$$y = 3x^2 + 1$$

$y \rightarrow$ dependent variable

$x \rightarrow$ independent variable

In the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (6)$$

It has one dependent variable V and two independent variables.

INTRODUCTION

Classifications of DE in terms of

- Type
- Order
- Linearity
- Homogeneity
- Autonomy

INTRODUCTION

Classifications by Type

1**. Ordinary Differential Equation (ODE)**

-If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable.

2. Partial Differential Equation (PDE)

- An equation containing partial derivatives of one or more dependent variables of two or more independent variables.

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Example 1.1.1 An Ordinary Differential Equation

Here's a typical elementary ODE, with some of its components indicated:

unknown function, y ↓

$$3\frac{dy}{dt} = y$$

independent variable, t ↑

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Other examples (ODE)

$$\frac{di}{dt} + 10i = 2\cos 4t$$

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

$$x''(t) - 5x'(t) + 6x(t) = 0$$

$$\frac{dx}{dt} = \frac{3t^2 + 4t + 2}{2(x-1)}$$

INTRODUCTION

Other examples (ODE)

$$\frac{dy}{dx} = \cos x \tag{1}$$

$$\frac{d^2y}{dx^2} + k^2y = 0 \tag{2}$$

$$\left(x^2 + y^2\right) dx - 2xy\, dy = 0 \tag{3}$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} = E\omega \sin \omega t \tag{5}$$

$$\left(\frac{d^2w}{dx^2}\right)^3 - xy\frac{dw}{dx} + w = 0 \tag{7}$$

$$\frac{d^3x}{dy^3} + x\frac{dx}{dy} - 4xy = 0 \tag{8}$$

$$\frac{d^2y}{dy^2} + 7\left(\frac{dy}{dx}\right)^3 - 8y = 0 \tag{9}$$

$$\frac{d^2y}{dt^2} + \frac{d^2x}{dt^2} = x \tag{10}$$

INTRODUCTION

Partial differential equations

- If we are dealing with functions of *several* variables and the derivatives involved are *partial* derivatives, then we have a **partial differential equation** (**PDE**

Examples of partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2xy$$

$$u_x + yu_y = u$$

$$u_{xx} + u_{yy} = \sin t$$

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

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Examples of partial differential equation

$$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{4}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{6}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \tag{11}$$

INTRODUCTION

Classifications by Order

- The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation.

For example, equation (1) is a 1st-order equation

$$\frac{dy}{dx} = \cos x$$

while equation (2) is a second-order equation

$$\frac{d^2y}{dx^2} + k^2y = 0 \qquad (2)$$

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The equations

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

$$(w')^2 + 2t^3w' - 4t^2w = 0$$

$$\frac{dx}{dt} = \frac{3t^2 + 4t + 2}{2(x-1)}$$

are all first-order differential equations because the highest derivative in each equation is the first derivative.

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First-order differential equations are written in the form
M(x; y)dx + N(x; y)dy = 0 for convenience.

In symbols, one can express an nth-order differential equation in one variable by the general form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0 \tag{12}$$

where F is areal-valued function of $n+2$ variable: x, y, y', …, $y^{(n)}$.For both practical and theoretical reasons it is assumed that it is possible to solve an ordinary differential equation in the form (12) uniquely for the highest derivative of $y^{(n)}$ in the term of the remaining $n+1$ variables. The differential equation:

$$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n)}\right) \tag{13}$$

where f is a real-valued continuous function, is referred to as the *normal form* of equation (12).

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General Form for a Second-Order ODE

If y is an unknown function of x, then the second-order ordinary differential equation $2\frac{d^2y}{dx^2} + e^x\frac{dy}{dx} = y + \sin x$ can be written as $2\frac{d^2y}{dx^2} + e^x\frac{dy}{dx} - y - \sin x = 0$ or as

$$\underbrace{2y'' + e^x y' - y - \sin x}_{F(x,y,y',y'')} = 0.$$

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The equations

$$x''(t) - 5x'(t) + 6x(t) = 0$$

$$\ddot{x} + 3t\,\dot{x} + 2x = \sin(\omega t)$$

are second-order equations.

INTRODUCTION

Classifications by Linearity

- ***Linear and nonlinear ordinary differential equations***

 - Another important way to categorize differential equations is in terms of whether they are *linear* or *nonlinear*

Linear

An nth-order differential equation is said to be linear if F is linear in y, y', ….y^n. This means that an nth-order differential equation is linear when it is in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_x(x)y' + a_0(x)y - g(x) = 0$$

or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (14)$$

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Two important cases of (14) are:

1st-order differential equation (n=1

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and second-order differential equation (n=2):

$$\alpha_2\ (x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

INTRODUCTION

A differential equation is *linear* if

- The *dependent* variable and its *derivatives* are of first degree (linear)
- All *coefficients* are constants or functions of the *independent* variable

Examples:

$$\frac{dy}{dt} + y = 0$$

$$x^2 y'' - (x-1)y' + 8y = \sin 3x$$

$$\frac{dy}{dx} + x^2 = 0$$

$$(y - x)\,dx + 4x\,dy = 0$$

$$y'' + 2y' + y = 0$$

$$\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

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Non-Linear

- A non-linear ordinary differential equation is simply one that is not linear. Non-linear functions of the dependent variable, or its derivatives, such as sin y or e^y, cannot appear in a linear equation.

Examples:

(1) $(1-y)y' + 2y = e^x$

(2) $\frac{d^2y}{dx^2} + \sin y = 0$

(3) $\frac{d^4y}{dx^4} + y^2 = 0$

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To determine if this differential equation is linear or non-linear, let's rewrite it in standard form:

$$(1 - y)\frac{dy}{dx} + 2y = e^x$$

This equation can be expanded and written as:

$$\frac{dy}{dx} - y\frac{dy}{dx} + 2y = e^x$$

We can see that the equation involves a term $y\frac{dy}{dx}$, which is the product of the dependent variable y and its derivative $\frac{dy}{dx}$.

INTRODUCTION

Criteria for Linearity:

A differential equation is linear if:

1. The dependent variable y and its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ appear to the first power.
2. There are no products of y and its derivatives.
3. The coefficients of y and its derivatives depend only on the independent variable x, not on y.

INTRODUCTION

Classifications by Homogeneity

1. **Homogeneous Differential Equations**

Definition:
A differential equation is homogeneous if it can be written such that all terms involve the dependent variable and its derivatives, with no term that is a function of the independent variable alone.

- The general form is:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

INTRODUCTION

Classifications by Homogeneity

2. **Non-Homogeneous Differential Equations**:

A non-homogeneous differential equation has terms that include functions of the independent variable alone (or constants), and the equation is not equal to zero.

The general form is:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where $g(x)$ is a non-zero function of the independent variable x, or a constant term.

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Example:

Homogeneous DE: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

Non-homogeneous DE:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$$

INTRODUCTION

Classifications by Homogeneity

Example 1: Linear Homogeneous DE

Consider the second-order linear homogeneous differential equation:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Explanation:

- **Order**: Second-order (highest derivative is $\frac{d^2y}{dx^2}$).
- **Linearity**: Linear (the equation is a linear combination of y and its derivatives).
- **Homogeneity**: Homogeneous (right-hand side is zero).

INTRODUCTION

Classifications by Homogeneity

Example 2: Non-Linear Homogeneous DE

Consider the following non-linear differential equation:

$$\frac{dy}{dx} = y^2$$

Explanation:

- **Order**: First-order (highest derivative is $\frac{dy}{dx}$).
- **Linearity**: Non-linear (involves y^2).
- **Homogeneity**: Homogeneous (right-hand side depends only on y and not on x).

INTRODUCTION

Classifications by Homogeneity

Example 1: Linear Non-Homogeneous DE

Consider the second-order linear non-homogeneous differential equation:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

Explanation:

- **Order**: Second-order.
- **Linearity**: Linear.
- **Homogeneity**: Non-homogeneous (right-hand side is e^x, which is a function of x).

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Classifications by Homogeneity

Example 2: Non-Linear Non-Homogeneous DE

Consider the first-order non-linear differential equation:

$$\frac{dy}{dx} = y^2 + x$$

Explanation:

- **Order**: First-order.
- **Linearity**: Non-linear (involves y^2).
- **Homogeneity**: Non-homogeneous (right-hand side includes x, an external term).

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Other examples classify if linear, non linear, homogenous, non-homogenous

1. $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

2. $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^x$$

3. $$\frac{dy}{dx} + y^2 = 0$$

4. $$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = \sin(x)$$

5. $$\frac{dy}{dx} = x^2 - y$$

INTRODUCTION

Classifications by Autonomy

Autonomy refers to whether a differential equation explicitly depends on the independent variable (often denoted as x or t).

Differential equations can be classified as **autonomous** or **non-autonomous** based on this dependency.

1. Autonomous Differential Equations

- **Definition**: A differential equation is **autonomous** if the independent variable (e.g., t or x) does not explicitly appear in the equation. In other words, the equation depends only on the dependent variable and its derivatives.

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Form: Autonomous differential equations can be written as:

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ is a function of y only, and t (the independent variable) does not explicitly appear in the function.

Example:

$$\frac{dy}{dt} = y^2 - 3y$$

This equation is autonomous because it depends only on the dependent variable y and not on the independent variable t.

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2. Non-Autonomous Differential Equations

•**Definition**: A differential equation is **non-autonomous** if the independent variable explicitly appears in the equation. This means the equation depends on both the dependent variable and the independent variable.

Form: Non-autonomous differential equations can be written as:

$$\frac{dy}{dt} = f(t, y)$$

where $f(t, y)$ is a function of both t (the independent variable) and y (the dependent variable).

Example:

$$\frac{dy}{dt} = t^2 + y^2$$

This equation is non-autonomous because it explicitly depends on the independent variable t.

INTRODUCTION

Other examples

1. $$\frac{d^2y}{dt^2} = -ky$$

4. $$\frac{dy}{dt} = y + t$$

2. $$\frac{dx}{dt} = \sin(x)$$

5. $$\frac{d^2x}{dt^2} = 3t + 5x$$

3. $$\frac{dy}{dx} = e^{-y}$$

6. $$\frac{dy}{dx} = x^2 + \cos(y)$$

Exercises

State whether the equation is ordinary or partial, linear or nonlinear, and give its order.

1. $\frac{d^2x}{dt^2} + k^2x = 0.$

2. $(x^2 + y^2)\,dx + 2xy\,dy = 0.$

3. $y''' - 3y' + 2y = 0.$

4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

5. $x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = c_1.$

6 $(x + y)\,dx + (3x^2 - 1)\,dy = 0.$

7 $\left(\frac{d^3w}{dx^3}\right)^2 - 2\left(\frac{dw}{dx}\right)^4 + yw = 0.$

8 $y'' + 2y' - 8y = x^2 + \cos x.$

9. $\frac{d^3x}{dy^3} + x\frac{dx}{dy} - 4xy = 0$

10. $y' = ty^2$

ASSIGNMENT

For each of the following differential equations, determine whether the equation is
a) Ordinary or Partial, b) Linear or Nonlinear, c)Homogeneous or Nonhomogeneous,
and d) autonomous or non-autonomous and finally e) identify the Order of the differential equation.

6 $(x + y)\,dx + (3x^2 - 1)\,dy = 0.$

7 $\left(\frac{d^3w}{dx^3}\right)^2 - 2\left(\frac{dw}{dx}\right)^4 + yw = 0.$

8 $y'' + 2y' - 8y = x^2 + \cos x.$

9. $\frac{d^3x}{dy^3} + x\frac{dx}{dy} - 4xy = 0$

10. $y' = ty^2$

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