CALCULUS 2

AREAS AND VOLUME/ WORK , MULTIPLE INTEGRALS

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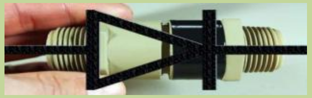
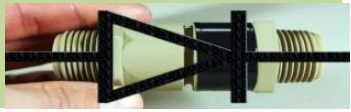


OBJECTIVES



- find the area between a curve $y = f(x)$ and an interval on the x-axis.
- find the area between two curves
- Learn solid of revolution, method of washers and disks
- Learn centroid, application of integrals
- Learn double and triple integrals

Partial Differentiation

$f(x, y) = x^2y + 3xy^2$

$f_x = \frac{\partial}{\partial x}(x^2y + 3xy^2)$

differentiating with respect to x, treating y as constant

$x^2y + xy^2$

$f_x = \frac{\partial}{\partial x}(x^2y + xy^2)$

$= y\frac{d}{dx}(x^2) + y^2\frac{d}{dx}(x)$

$= y(2x) + y^2(1)$

$f_x = 2xy + y^2$

$x^2y + xy^2$

$f_y = \frac{\partial}{\partial y}(x^2y + xy^2)$

$= x^2\frac{d}{dy}(y) + x\frac{d}{dy}(y^2)$

$= x^2(1) + x(2y)$

$= x^2 + 2y$

✅ Problem:

Find the area of the region bounded:

- **Above by**: $y = x + 7$
- **Below by**: $y = x^2$
- **Vertically by**: $x = 0$ and $x = 3$

$$A = \int_0^3 \left[(x+7) - x^2\right]\, dx$$

$$A = \int_0^3 (-x^2 + x + 7)\, dx$$

$$A = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 7x\right]_0^3$$

$$= -\frac{27}{3} + \frac{9}{2} + 21 = -9 + 4.5 + 21 = 16.5$$

$$\boxed{16.5 \text{ square units}}$$

Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$
over the interval $[0, 3]$ is revolved about the x-axis.

$$V = \pi \int_0^2 \left[(f(x))^2 - (g(x))^2\right] dx$$

$$V = \pi \int_0^3 \left[\left(\frac{1}{2} + x^2\right)^2 - x^2\right] dx$$

$$\frac{1}{4} + x^2 + x^4 \quad - x^2$$

$$V = \pi \int_0^3 \left(\frac{1}{4} + x^4\right) dx$$

$$= \pi \left[\frac{1}{4}x + \frac{x^5}{5}\right]_0^3 = \pi \left(\frac{3}{4} + \frac{243}{5}\right)$$

$$\boxed{V = \frac{987\pi}{20} \text{ cubic units}}$$

Iterated Integrals

$$f(x, y) = \int f_x(x, y)\,dx \qquad \text{Integrate with respect to } x.$$

$$= \int 2xy\,dx \qquad \text{Hold } y \text{ constant.}$$

$$= y\int 2x\,dx \qquad \text{Factor out constant } y.$$

$$= y(x^2) + C(y) \qquad \text{Antiderivative of } 2x \text{ is } x^2.$$

$$= x^2y + C(y). \qquad C(y) \text{ is a function of } y.$$

The "constant" of integration, $C(y)$, is a function of y. In other words, by integrating with respect to x, you are able to recover $f(x, y)$ only partially. The total recovery of a function of x and y from its partial derivatives is a topic you will study in Chapter 15. For now, you will focus on extending definite integrals to functions of several variables. For instance, by considering y constant, you can apply the Fundamental Theorem of Calculus to evaluate

$$\int_1^{2y} 2xy\,dx = x^2y\Big]_1^{2y} = (2y)^2y - (1)^2y = 4y^3 - y.$$

x is the variable of integration and y is fixed.

Replace x by the limits of integration.

The result is a function of y.

Similarly, you can integrate with respect to y by holding x fixed. Both procedures are summarized as follows.

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y)\,dx = f(x, y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \qquad \text{With respect to } x$$

$$\int_{g_1(x)}^{g_2(x)} f_y(x, y)\,dy = f(x, y)\Big]_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \qquad \text{With respect to } y$$

EXAMPLE 1 **Integrating with Respect to *y***

Evaluate $\int_1^x (2xy + 3y^2)\,dy$.

Solution Considering x to be constant and integrating with respect to y, you have

$$\int_1^x (2xy + 3y^2)\,dy = \Big[xy^2 + y^3\Big]_1^x \qquad \text{Integrate with respect to } y.$$
$$= (2x^3) - (x + 1)$$
$$= 2x^3 - x - 1.$$

Notice in Example 1 that the integral defines a function of x and can *itself* be integrated, as shown in the next example.

EXAMPLE 2 **The Integral of an Integral**

Evaluate $\int_1^2 \left[\int_1^x (2xy + 3y^2)\,dy\right] dx$.

Solution Using the result of Example 1, you have

$$\int_1^2 \left[\int_1^x (2xy + 3y^2)\,dy\right] dx = \int_1^2 (2x^3 - x - 1)\,dx$$
$$= \left[\frac{x^4}{2} - \frac{x^2}{2} - x\right]_1^2 \qquad \text{Integrate with respect to } x.$$
$$= 4 - (-1)$$
$$= 5.$$

EXAMPLE 4 Finding Area by an Iterated Integral

Use an iterated integral to find the area of the region bounded by the graphs of

$$f(x) = \sin x$$ Sine curve forms upper boundary.

and

$$g(x) = \cos x$$ Cosine curve forms lower boundary.

between $x = \pi/4$ and $x = 5\pi/4$.

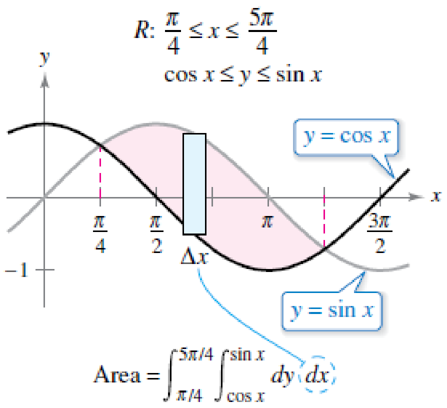


Figure 14.5

Solution Because f and g are given as functions of x, a vertical representative rectangle is convenient, and you can choose $dy\,dx$ as the order of integration, as shown in Figure 14.5. The outside limits of integration are

$$\frac{\pi}{4} \le x \le \frac{5\pi}{4}.$$

Moreover, because the rectangle is bounded above by $f(x) = \sin x$ and below by $g(x) = \cos x$, you have

$$\text{Area of } R = \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy\,dx$$

$$= \int_{\pi/4}^{5\pi/4} y \Big]_{\cos x}^{\sin x} dx$$ Integrate with respect to y.

$$= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$ Integrate with respect to x.

$$= 2\sqrt{2}.$$

EXAMPLE 5 Comparing Different Orders of Integration

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 dx\, dy.$$

Then find another iterated integral using the order $dy\, dx$ to represent the same area and show that both integrals yield the same value.

Solution From the given limits of integration, you know that

$$y^2 \le x \le 4 \qquad \text{Inner limits of integration}$$

which means that the region R is bounded on the left by the parabola $x = y^2$ and on the right by the line $x = 4$. Furthermore, because

$$0 \le y \le 2 \qquad \text{Outer limits of integration}$$

you know that R is bounded below by the x-axis, as shown in Figure 14.6(a). The value of this integral is

$$\int_0^2 \int_{y^2}^4 dx\, dy = \int_0^2 x \Big]_{y^2}^4 dy \qquad \text{Integrate with respect to } x.$$

$$= \int_0^2 (4 - y^2)\, dy$$

$$= \left[4y - \frac{y^3}{3}\right]_0^2 \qquad \text{Integrate with respect to } y.$$

$$= \frac{16}{3}.$$

11. $\int_0^1 \int_0^2 (x + y)\, dy\, dx$ **12.** $\int_{-1}^1 \int_{-2}^2 (x^2 - y^2)\, dy\, dx$

Problem 11:

$$\int_0^1 \int_0^2 (x + y)\, dy\, dx$$

Step 1: Integrate with respect to y:

$$\int_0^1 \left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^1 \left(2x + \frac{4}{2}\right) dx = \int_0^1 (2x + 2)\, dx$$

Step 2: Integrate with respect to x:

$$\left[x^2 + 2x\right]_0^1 = (1 + 2) - 0 = \boxed{3}$$

Problem 12:

$$\int_{-1}^{1}\int_{-2}^{2}(x^2-y^2)\,dy\,dx$$

Note: This is a symmetric region over both axes. x^2 is even, y^2 is even, so x^2-y^2 is **even in both variables**, but we integrate **odd in** y part.

Let's solve directly:

Step 1: Integrate with respect to y:

$$\int_{-1}^{1}\left[x^2y-\frac{y^3}{3}\right]_{-2}^{2}dx=\int_{-1}^{1}\left(2x^2-\frac{8}{3}-(-2x^2+\frac{8}{3})\right)dx=\int_{-1}^{1}(4x^2-\frac{16}{3})dx$$

Step 2: Integrate with respect to x:

$$\int_{-1}^{1}\left(4x^2-\frac{16}{3}\right)dx=4\int_{-1}^{1}x^2dx-\frac{16}{3}\int_{-1}^{1}dx$$

$$=4\left[\frac{x^3}{3}\right]_{-1}^{1}-\frac{16}{3}(2)=4\cdot\left(\frac{1}{3}-(-\frac{1}{3})\right)-\frac{32}{3}=4\cdot\frac{2}{3}-\frac{32}{3}=\frac{8}{3}-\frac{32}{3}=\boxed{-\frac{24}{3}=-8}$$

DOUBLE INTEGRALS

$$\text{Volume} = \iint\limits_R f(x, y)\, dA$$

EXAMPLE

$$\iint\limits_R 6y\sqrt{x} - 2y^3\, dA \qquad R = [1, 4] \times [0, 3]$$

INTEGRAL CALCULUS

DOUBLE INTEGRALS

In Problems 2–6, evaluate the integral on the left.

2. $\int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 (x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right] = \frac{1}{6}$

3. $\int_1^2 \int_y^{3y} (x + y)dx\,dy = \int_1^2 [\frac{1}{2}x^2 + xy]_y^{3y}\,dy = \int_1^2 6y^2 dy = [2y^3]_1^2 = 14$

4. $\int_{-1}^2 \int_{2x^2-2}^{x^2+x} x\,dy\,dx = \int_{-1}^2 [xy]_{2x^2-2}^{x^2+x}\,dx = \int_{-1}^2 (x^3 + x^2 - 2x^3 + 2x)\,dx = \frac{9}{4}$

5. $\int_0^\pi \int_0^{\cos\theta} \rho\sin\theta\,d\rho\,d\theta = \int_0^\pi [\frac{1}{2}\rho^2\sin\theta]_0^{\cos\theta}\,d\theta = \frac{1}{2}\int_0^\pi \cos^2\theta\sin\theta\,d\theta = [-\frac{1}{6}\cos^3\theta]_0^\pi = \frac{1}{3}$

SOLUTIONS

$$\iint\limits_R 6y\sqrt{x} - 2y^3\,dA = \int_0^3 \int_1^4 6yx^{\frac{1}{2}} - 2y^3\,dx\,dy$$

$$= \int_0^3 \left(4yx^{\frac{3}{2}} - 2xy^3\right)\Big|_1^4 dy = \int_0^3 28y - 6y^3\,dy$$

$$: \int_0^3 28y - 6y^3\,dy = \left(14y^2 - \frac{3}{2}y^4\right)\Big|_0^3 = \boxed{\frac{9}{2}}$$

INTEGRAL CALCULUS

EXAMPLE 2

$$= \int_{-1}^{4}\int_{2}^{3} 12x - 18y\,dy\,dx$$

$$= \int_{-1}^{4} (12xy - 9y^2)\Big|_2^3\,dx = \int_{-1}^{4} 12x - 45\,dx$$

$$\int_{-1}^{4} 12x - 45\,dx = (6x^2 - 45x)\Big|_{-1}^{4} = \boxed{-135}$$

Problem:

Evaluate the double integral $\iint_R (2x+3y)\,dA$, where R is the rectangular region given by $0 \le x \le 1$ and $0 \le y \le 2$.

$$\int_0^1 \int_0^2 (2x+3y)\,dy\,dx$$

$$\int_0^2 (2x+3y)\,dy = \left[2xy + \frac{3y^2}{2}\right]_0^2$$

$$\left[2x \cdot 2 + \frac{3 \cdot 2^2}{2}\right] - \left[2x \cdot 0 + \frac{3 \cdot 0^2}{2}\right] = [4x+6]$$

$$\int_0^1 (4x+6)\,dx = \left[2x^2 + 6x\right]_0^1$$

$$[2 \cdot 1^2 + 6 \cdot 1] - [2 \cdot 0^2 + 6 \cdot 0] = [2+6] - [0] = 8$$

$$\int_0^1 \int_0^x (x^2 + y^2)\, dy\, dx$$

$$\int_0^x (x^2 + y^2)\, dy = \left[x^2 y + \frac{y^3}{3}\right]_0^x$$

$$\left[x^3 + \frac{x^3}{3}\right] - \left[x^2 \cdot 0 + \frac{0^3}{3}\right] = \left[x^3 + \frac{x^3}{3}\right] = x^3 + \frac{x^3}{3} = \frac{4x^3}{3}$$

$$\int_0^1 \frac{4x^3}{3}\, dx = \frac{4}{3} \int_0^1 x^3\, dx = \frac{4}{3} \left[\frac{x^4}{4}\right]_0^1$$

$$\frac{4}{3} \left[\frac{1^4}{4} - \frac{0^4}{4}\right] = \frac{4}{3} \left[\frac{1}{4}\right] = \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3}$$

Integrating with respect to y

$$\int_1^x (2xy + 3y^2)\,dy$$

$$2x\frac{y^2}{2} + \frac{3y^3}{3}\Big|_1^x$$

$$xy^2 + y^3\Big|_1^x$$

$$= [x(x^2) + x^3] - [x(1^2) + 1^3]$$

$$= (x^3 + x^3] - (x+1)$$

$$\boxed{= 2x^3 - x - 1}$$

Integral of integral / iterated integral

$$= \int_1^2 \left[\int_1^x (2x + 3y^2)\,dy\right] dx$$

$$= \int_1^2 (2x^3 - x - 1)\,dx$$

$$= \frac{2x^4}{4} - \frac{x^2}{2} - x\Big|_1^2$$

$$= (8 - 2 - 2) - \left(\frac{2}{4} - \frac{1}{2} - 1\right)$$

$$= 4 - (-1)$$

$$\boxed{= 5}$$

Triple Integrals

$$\iiint\limits_E f(x, y, z)\, dV$$

Let's start simple by integrating over the box,

$$B = [a, b] \times [c, d] \times [r, s]$$

Note that when using this notation we list the x's first, the y's second and the z's third.

The triple integral in this case is,

$$\iiint\limits_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

INTEGRAL CALCULUS

Example 1 Evaluate the following integral.

$$\iiint\limits_B 8xyz\,dV \qquad B = [2,3] \times [1,2] \times [0,1]$$

$$\begin{aligned}
\iiint\limits_B 8xyz\,dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\
&= \int_1^2 \int_2^3 4xyz^2\Big|_0^1 \,dx\,dy \\
&= \int_1^2 \int_2^3 4xy\,dx\,dy \\
&= \int_1^2 2x^2y\Big|_2^3 \,dy \\
&= \int_1^2 10y\,dy = 15
\end{aligned}$$

. Evaluate $\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2y - z^3\,dz\,dy\,dx$

$$\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2y - z^3\,dz\,dy\,dx = \int_{2}^{3}\int_{-1}^{4} \left(4x^2yz - \frac{1}{4}z^4\right)\Big|_{1}^{0}\,dy\,dx$$

$$= \int_{2}^{3}\int_{-1}^{4} \frac{1}{4} - 4x^2y\,dy\,dx$$

INTEGRAL CALCULUS

$$\int_2^3 \int_{-1}^4 \int_1^0 4x^2y - z^3 \, dz \, dy \, dx = \int_2^3 \left(\frac{1}{4}y - 2x^2y^2 \right) \Big|_{-1}^4 \, dx$$

$$= \int_2^3 \frac{5}{4} - 30x^2 \, dx$$

$$\int_2^3 \int_{-1}^4 \int_1^0 4x^2y - z^3 \, dz \, dy \, dx = \left(\frac{5}{4}x - 10x^3 \right) \Big|_2^3 = \boxed{-\frac{755}{4}}$$

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where c is any real number (see Exercise 107).

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

Partial Differentiation

$$f(x,y) = x^2y + 3xy^2$$

$$f_x = \frac{\partial}{\partial x}(x^2y + 3xy^2)$$

differentiating with respect to x, treating y as constant

$$x^2y + xy^2$$

$$f_x = \frac{\partial}{\partial x}(x^2y + xy^2)$$

$$= y\frac{d}{dx}(x^2) + y^2\frac{d}{dx}(x)$$

$$= y(2x) + y^2(1)$$

$$f_x = 2xy + y^2$$

$$x^2y + xy^2$$

$$f_y = \frac{\partial}{\partial y}(x^2y + xy^2)$$

$$= x^2\frac{d}{dy}(y) + x\frac{d}{dy}(y^2)$$

$$= x^2(1) + x(2y)$$

$$= x^2 + 2y$$

Integrating with respect to y

$$\int_1^x (2xy + 3y^2)\,dy$$

$$2x\frac{y^2}{2} + \frac{3y^3}{3}\Big|_1^x$$

$$xy^2 + y^3\Big|_1^x$$

$$= [x(x^2) + x^3] - [x(1^2) + 1^3]$$

$$= [x^3 + x^3] - (x+1)$$

$$= 2x^3 - x - 1$$

Integral of integral / iterated integral

$$= \int_1^2 \left[\int_1^x (2x + 3y^2)\,dy\right] dx$$

$$= \int_1^2 (2x^3 - x - 1)\,dx$$

$$= \frac{2x^4}{4} - \frac{x^2}{2} - x\Big|_1^2$$

$$= (8 - 2 - 2) - \left(\frac{2}{4} - \frac{1}{2} - 1\right)$$

$$= 4 - (-1)$$

$$= 5$$

3) $\int_0^1 \int_{x^2}^{x} dy\, dx$

$= \int_0^1 \left[\int_{x^2}^{x} dy \right] dx$

$= \int_0^1 \left(y \Big|_{x^2}^{x} \right) dx$

$= \int_0^1 (x - x^2)\, dx$

$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \left(\frac{1}{2} - \frac{1}{3} \right) - (0)$

$\boxed{= \frac{1}{6}}$

4) $\int_{-1}^{4} \int_{2}^{3} (12x - 18y)\, dy\, dx$

$\int_{-1}^{4} \left[\int_2^3 (12x - 18y)\, dy \right] dx$

$\int_{-1}^{4} \left(12xy - 9y^2 \Big|_2^3 \right) dx$

$\int_{-1}^{4} (36x - 81) - (24x - 36)\, dx$

$\int_{-1}^{4} (36x - 24x - 81 + 36)\, dx$

$\int_{-1}^{4} (12x - 45)\, dx = \frac{12x^2}{2} - 45x \Big|_{-1}^{4}$

$= 6x^2 - 45x$

$= (-84) - (51) = \boxed{-135}$

$$\int_2^3 \int_{-1}^4 \int_1^0 dz\,dy\,dx$$

$$\int_2^3 \int_{-1}^4 \left[\int_1^0 dz \right] dy\,dx$$

$$\int_2^3 \int_{-1}^4 \left(z \Big|_1^0 \right) dy\,dx$$

$$\int_2^3 \int_{-1}^4 (0-1)\,dy\,dx$$

$$\int_2^3 \int_{-1}^4 -dy\,dx$$

$$\int_2^3 \left[\int_{-1}^4 -dy \right] dx$$

$$\int_2^3 \left(-y \Big|_{-1}^4 \right) dx$$

$$\int_2^3 (-4)-(1)\,dx$$

$$\int_2^3 -5\,dx$$

$$\int_2^3 -5\,dx$$

$$-5x \Big|_2^3$$

$$(-15)-(-10)$$

$$-15+10$$

$$\boxed{= -5}$$

$$\int_2^3\int_{-1}^4\int_1^0 (4x^2y - z^3)\,dz\,dy\,dx$$

$$\int_2^3\int_{-1}^4\left[\int_1^0 (4x^2y - z^3)\,dz\right]dy\,dx$$

$$\int_2^3\int_{-1}^4\left[4x^2yz - \frac{z^4}{4}\right]_1^0 dy\,dx$$

$$\int_2^3\int_{-1}^4 0 - \left[4x^2y(1) - \frac{1}{4}\right]dy\,dx$$

$$\int_2^3\int_{-1}^4 \left(-4x^2y + \frac{1}{4}\right)dy\,dx$$

$$\int_2^3\left[\int_{-1}^4 \left(-4x^2y + \frac{1}{4}\right)dy\right]dx$$

$$\int_2^3 -2x^2y^2 + \frac{1}{4}y\Big|_{-1}^4 \,dx$$

$$\int_2^3 \left[\left(-32x^2 + 1\right) - \left(-2x^2 - \frac{1}{4}\right)\right]dx$$

$$\int_2^3 \left(-30x^2 + \frac{5}{4}\right)dx$$

$$-10x^3 + \frac{5}{4}x\Big|_2^3 = \left(-270 + \frac{15}{4}\right) - \left(-80 + \frac{10}{4}\right)$$

$$= \frac{-755}{4}$$

IMPROPER INTEGRALS

- Improper integrals with infinite intervals of integration:

$$\int_{1}^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x \, dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_{1}^{2} \frac{dx}{x-1}, \quad \int_{0}^{\pi} \tan x \, dx$$

- Improper integrals with infinite discontinuities and infinite intervals of integration:

$$\int_{0}^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_{1}^{+\infty} \sec x \, dx$$

7.8.1 DEFINITION The *improper integral of f over the interval* $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

7.8.3 DEFINITION The *improper integral of f over the interval* $(-\infty, b]$ is defined to be

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \tag{2}$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not.

The *improper integral of f over the interval* $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^{+\infty} f(x)\,dx \tag{3}$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

EXAMPLE 1 An Improper Integral That Diverges

Evaluate $\int_1^b \frac{dx}{x}$.

Solution

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} \qquad \text{Take limit as } b \to \infty.$$

$$= \lim_{b\to\infty} \Big[\ln x \Big]_1^b \qquad \text{Apply Log Rule.}$$

$$= \lim_{b\to\infty} (\ln b - 0) \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \infty \qquad \text{Evaluate limit.}$$

EXAMPLE 2 Improper Integrals That Converge

Evaluate each improper integral.

a. $\int_0^\infty e^{-x}\,dx$

b. $\int_0^\infty \frac{1}{x^2+1}\,dx$

Solution

a.
$$\begin{aligned}\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-x}\,dx\\ &= \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b\\ &= \lim_{b\to\infty}\left(-e^{-b}+1\right)\\ &= 1\end{aligned}$$

b.
$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx\\ &= \lim_{b\to\infty}\Big[\arctan x\Big]_0^b\\ &= \lim_{b\to\infty}\arctan b\\ &= \frac{\pi}{2}\end{aligned}$$

Example 1 Evaluate

$$\text{(a)} \int_1^{+\infty} \frac{dx}{x^3} \qquad \text{(b)} \int_1^{+\infty} \frac{dx}{x}$$

Solution (a). Following the definition, we replace the infinite upper limit by a finite upper limit b, and then take the limit of the resulting integral. This yields

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^3} = \lim_{b \to +\infty} \left[-\frac{1}{2x^2} \right]_1^b = \lim_{b \to +\infty} \left(\frac{1}{2} - \frac{1}{2b^2} \right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is $1/2$.

Solution (b).

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} [\ln x]_1^b = \lim_{b \to +\infty} \ln b = +\infty$$

In this case the integral diverges and hence has no value. ◄

Basic Operations

1. **Addition and Subtraction:**

   - $\infty + a = \infty$ (for any finite a)
   - $\infty - a = \infty$ (for any finite a)
   - $\infty + \infty = \infty$
   - $\infty - \infty$ is indeterminate (undefined)

2. **Multiplication:**

   - $\infty \times a = \infty$ (for any positive finite a)
   - $\infty \times (-a) = -\infty$ (for any positive finite a)
   - $\infty \times \infty = \infty$
   - $\infty \times 0$ is indeterminate (undefined)

3. **Division:**

   - $\frac{\infty}{a} = \infty$ (for any positive finite a)
   - $\frac{\infty}{-\infty} = -\infty$
   - $\frac{a}{\infty} = 0$ (for any finite a)
   - $\frac{\infty}{\infty}$ is indeterminate (undefined)

Exponential Functions

1. **Exponential Growth:**

   - $e^{\infty} = \infty$
   - $e^{-\infty} = 0$

2. **Power Functions:**

   - $a^{\infty} = \infty$ (for any $a > 1$)
   - $a^{-\infty} = 0$ (for any $a > 1$)
   - 0^{∞} and 1^{∞} are indeterminate (und

Logarithmic Functions

1. **Logarithms:**

   - $\ln(\infty) = \infty$
   - $\ln(0) = -\infty$

Trigonometric Functions

1. **Arctangent:**

   - $\arctan(\infty) = \frac{\pi}{2}$
   - $\arctan(-\infty) = -\frac{\pi}{2}$

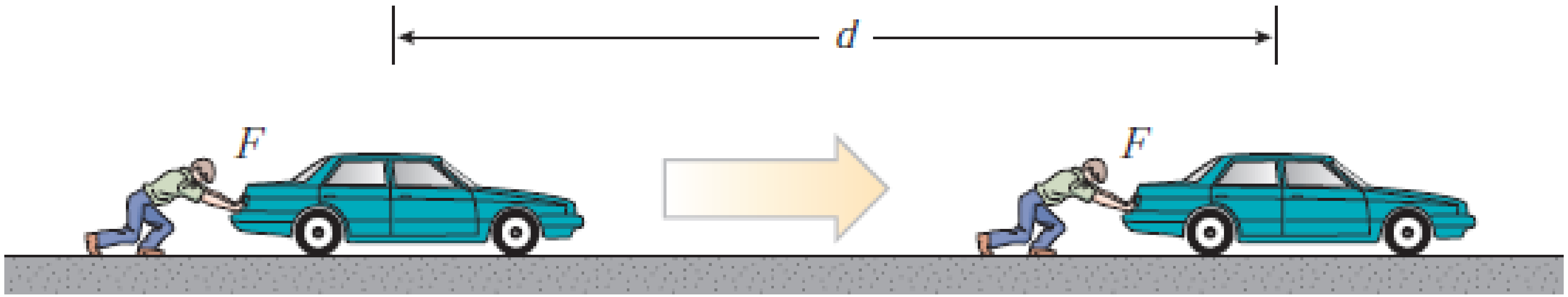
Limits

1. **Limits Involving Infinity:**

   - $\lim_{x \to \infty} f(x) = L$ means $f(x)$ approaches L as x approaches infinity.
   - $\lim_{x \to -\infty} f(x) = L$ means $f(x)$ approaches L as x approaches negative infinity.
   - Common indeterminate forms include $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^{∞}.

WORK DONE BY A CONSTANT FORCE APPLIED IN THE DIRECTION OF MOTION

When a stalled car is pushed, the speed that the car attains depends on the force F with which it is pushed and the distance d over which that force is applied (Figure 6.6.1). Force and distance appear in the following definition of work.



▶ **Figure 6.6.1**

6.6.1 DEFINITION If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d, then we define the *work* W performed by the force on the object to be

$$W = F \cdot d \tag{1}$$

Common units for measuring force are newtons (N) in the International System of Units (SI), dynes (dyn) in the centimeter-gram-second (CGS) system, and pounds (lb) in the British Engineering (BE) system. One newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2, one dyne is the force required to give a mass of 1 g an acceleration of 1 cm/s^2, and one pound of force is the force required to give a mass of 1 slug an acceleration of 1 ft/s^2.

It follows from Definition 6.6.1 that work has units of force times distance. The most common units of work are newton-meters (N·m), dyne-centimeters (dyn·cm), and foot-pounds (ft·lb). As indicated in Table 6.6.1, one newton-meter is also called a *joule* (J), and one dyne-centimeter is also called an *erg*. One foot-pound is approximately 1.36 J.

Table 6.6.1

SYSTEM	FORCE	×	DISTANCE	=	WORK
SI	newton (N)		meter (m)		joule (J)
CGS	dyne (dyn)		centimeter (cm)		erg
BE	pound (lb)		foot (ft)		foot-pound (ft·lb)

CONVERSION FACTORS:

$1\ \text{N} = 10^5\ \text{dyn} \approx 0.225\ \text{lb}$	$1\ \text{lb} \approx 4.45\ \text{N}$
$1\ \text{J} = 10^7\ \text{erg} \approx 0.738\ \text{ft·lb}$	$1\ \text{ft·lb} \approx 1.36\ \text{J} = 1.36 \times 10^7\ \text{erg}$

WORK DONE BY A VARIABLE FORCE APPLIED IN THE DIRECTION OF MOTION

6.6.3 DEFINITION Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the *work* W performed by the force on the object to be

$$W = \int_a^b F(x)\,dx \qquad (2)$$

Hooke's law [Robert Hooke (1635–1703), English physicist] states that under appropriate conditions a spring that is stretched x units beyond its natural length pulls back with a force

$$F(x) = kx$$

where k is a constant (called the *spring constant* or *spring stiffness*). The value of k depends on such factors as the thickness of the spring and the material used in its composition. Since $k = F(x)/x$, the constant k has units of force per unit length.

▶ **Example 3** A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

(a) Find the spring constant k.

(b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a). From Hooke's law,

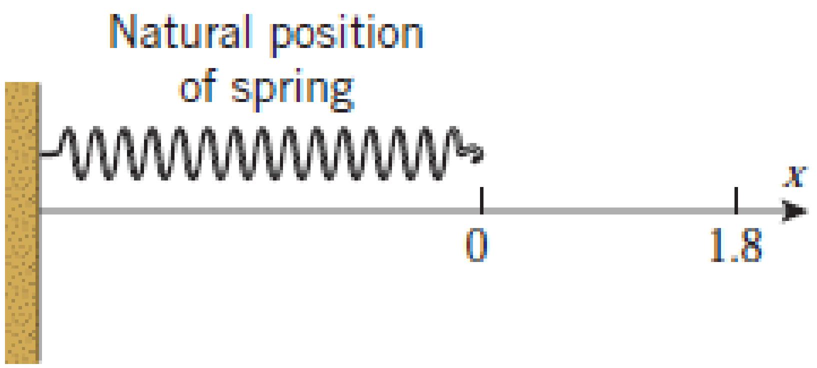
$$F(x) = kx$$

From the data, $F(x) = 5$ N when $x = 1$ m, so $5 = k \cdot 1$. Thus, the spring constant is $k = 5$ newtons per meter (N/m). This means that the force $F(x)$ required to stretch the spring x meters is

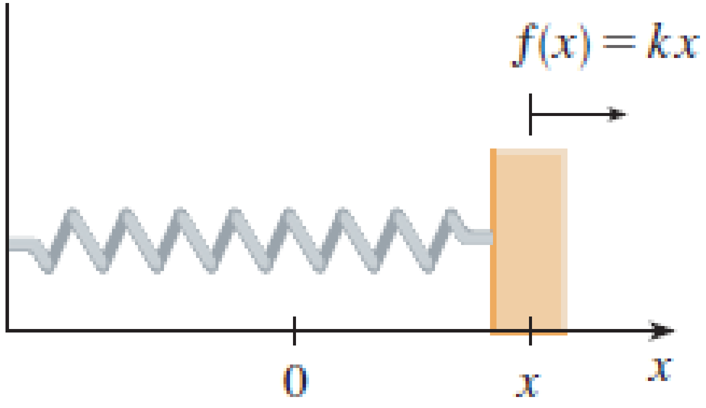
$$F(x) = 5x \tag{3}$$

Solution (b). Place the spring along a coordinate line as shown in Figure 6.6.3. We want to find the work W required to stretch the spring over the interval from $x = 0$ to $x = 1.8$. From (2) and (3) the work W required is

$$W = \int_a^b F(x)\,dx = \int_0^{1.8} 5x\,dx = \left.\frac{5x^2}{2}\right]_0^{1.8} = 8.1 \text{ J} \blacktriangleleft$$



▲ **Figure 6.6.3**



(b) Stretched position of spring

EXAMPLE 4 A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

SOLUTION According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$. When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that $f(0.05) = 40$, so

$$0.05k = 40 \qquad k = \frac{40}{0.05} = 800$$

Thus $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x\,dx = 800 \frac{x^2}{2}\Big]_{0.05}^{0.08}$$

$$= 400[(0.08)^2 - (0.05)^2] = 1.56 \text{ J} \qquad \square$$

EXAMPLE 5. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

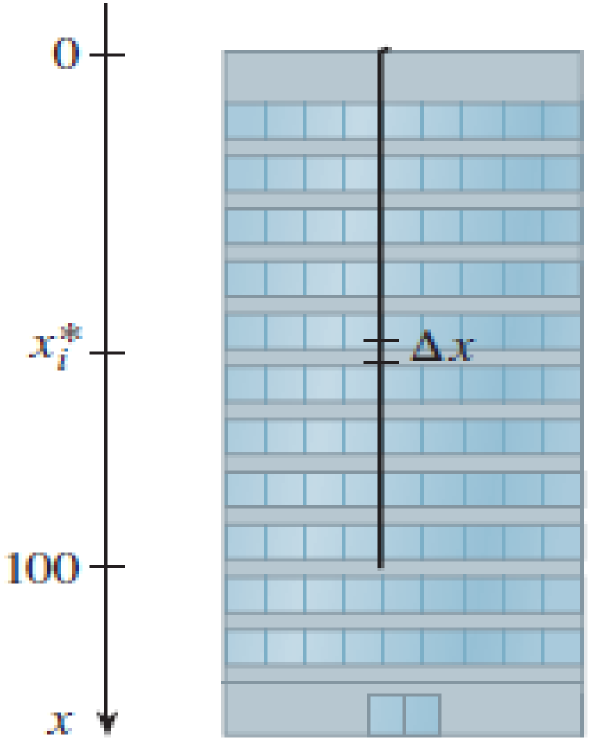


FIGURE 2

SOLUTION Here we don't have a formula for the force function, but we can use an argument similar to the one that led to Definition 4.

Let's place the origin at the top of the building and the x-axis pointing downward as in Figure 2. We divide the cable into small parts with length Δx. If x_i^* is a point in the ith such interval, then all points in the interval are lifted by approximately the same amount, namely x_i^*. The cable weighs 2 pounds per foot, so the weight of the ith part is $2\Delta x$. Thus the work done on the ith part, in foot-pounds, is

$$\underbrace{(2\Delta x)}_{\text{force}} \quad \underbrace{x_i^*}_{\text{distance}} = 2x_i^* \, \Delta x$$

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \to 0$):

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 2x_i^* \Delta x = \int_0^{100} 2x \, dx$$

$$= x^2 \Big]_0^{100} = 10{,}000 \text{ ft-lb}$$

■ If we had placed the origin at the bottom of the cable and the x-axis upward, we would have gotten

$$W = \int_0^{100} 2(100 - x) \, dx$$

which gives the same answer.

EXAMPLE 5 A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

SOLUTION Let's measure depths from the top of the tank by introducing a vertical coordinate line as in Figure 3. The water extends from a depth of 2 m to a depth of 10 m and so we divide the interval [2, 10] into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the ith subinterval. This divides the water into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δx. We can compute r_i from similar triangles, using Figure 4, as follows:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \qquad r_i = \tfrac{2}{5}(10 - x_i^*)$$

Thus an approximation to the volume of the ith layer of water is

$$V_i \approx \pi r_i^2 \, \Delta x = \frac{4\pi}{25}(10 - x_i^*)^2 \, \Delta x$$

and so its mass is

$$\begin{aligned} m_i &= \text{density} \times \text{volume} \\ &\approx 1000 \cdot \frac{4\pi}{25}(10 - x_i^*)^2 \, \Delta x = 160\pi(10 - x_i^*)^2 \, \Delta x \end{aligned}$$

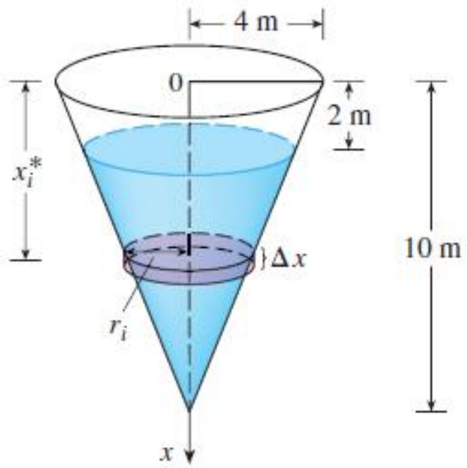
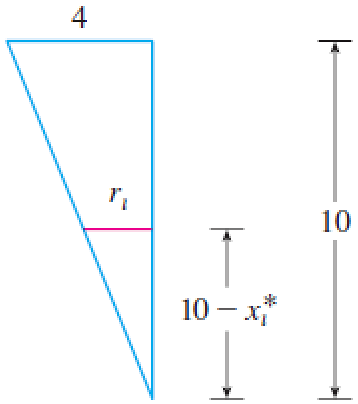


FIGURE 3



Each particle in the layer must travel a distance of approximately x_i^*. The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^*:

$$W_i \approx F_i x_i^* \approx 1570\pi x_i^*(10 - x_i^*)^2\,\Delta x$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1570\pi x_i^*(10 - x_i^*)^2\,\Delta x = \int_2^{10} 1570\pi x(10 - x)^2\,dx$$

$$= 1570\pi \int_2^{10} (100x - 20x^2 + x^3)\,dx = 1570\pi\left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4}\right]_2^{10}$$

$$= 1570\pi\left(\tfrac{2048}{3}\right) \approx 3.4 \times 10^6\ \text{J}$$

□

Centroids of Plane Areas

- **THE MASS OF A PHYSICAL BODY** is a measure of the quantity of matter in it, whereas the volume of the body is a measure of the space it occupies.
- If the mass per unit volume is the same throughout, the body is said to be ***homogeneous*** or to have ***constant density.***
- It is highly desirable in physics and mechanics to consider a given mass as concentrated at a point, called its center of mass (also, its center of gravity).
- For a homogeneous body, this point coincides with its geometric center or ***centroid.*** For example, the center of mass of a homogeneous rubber ball coincides with the centroid (center) of the ball considered as a geometric solid (a sphere).
- The centroid of a rectangular sheet of paper lies midway between the two surfaces but it may well be considered as located on one of the surfaces at the intersection of the diagonals. Then the center of mass of a thin sheet coincides with the centroid of the sheet considered as a plane area.

Center of Mass in a One-Dimensional System

You will now consider two types of moments of a mass—the **moment about a point** and the **moment about a line.** To define these two moments, consider an idealized situation in which a mass m is concentrated at a point. If x is the distance between this point mass and another point P, then the **moment of m about the point P is**

$$\text{Moment} = mx$$

and x is the **length of the moment arm.**

The concept of moment can be demonstrated simply by a seesaw, as shown in Figure 7.53. A child of mass 20 kilograms sits 2 meters to the left of fulcrum P, and an older child of mass 30 kilograms sits 2 meters to the right of P. From experience, you know that the seesaw will begin to rotate clockwise, moving the larger child down. This rotation occurs because the moment produced by the child on the left is less than the moment produced by the child on the right.

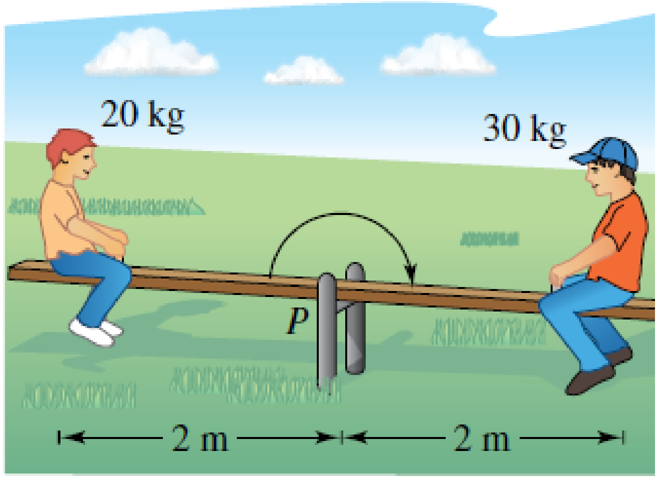
$$\text{Left moment} = (20)(2) = 40 \text{ kilogram-meters}$$
$$\text{Right moment} = (30)(2) = 60 \text{ kilogram-meters}$$

To balance the seesaw, the two moments must be equal. For example, if the larger child moved to a position $\frac{4}{3}$ meters from the fulcrum, then the seesaw would balance, because each child would produce a moment of 40 kilogram-meters.

To generalize this, you can introduce a coordinate line on which the origin corresponds to the fulcrum, as shown in Figure 7.54. Several point masses are located on the x-axis. The measure of the tendency of this system to rotate about the origin is the **moment about the origin,** and it is defined as the sum of the n products $m_i x_i$. The moment about the origin is denoted by M_0 and can be written as

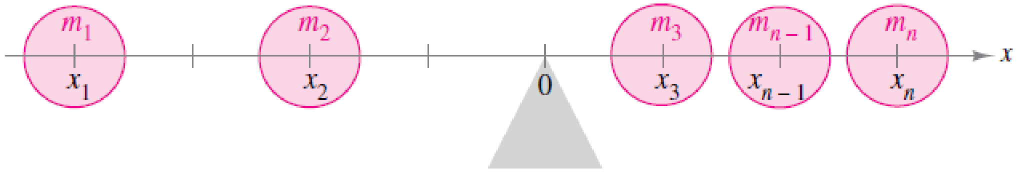
$$M_0 = m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n.$$

If M_0 is 0, then the system is said to be in **equilibrium.**



The seesaw will balance when the left and the right moments are equal.

Figure 7.53



If $m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n = 0$, then the system is in equilibrium.

Figure 7.54

For a system that is not in equilibrium, the **center of mass** is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium. If the system were translated $\bar{x}$ units, then each coordinate x_i would become

$$(x_i - \bar{x})$$

and because the moment of the translated system is 0, you have

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = \sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \bar{x} = 0.$$

Solving for $\bar{x}$ produces

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about origin}}{\text{total mass of system}}.$$

When $m_1 x_1 + m_2 x_2 + \cdot \cdot \cdot + m_n x_n = 0$, the system is in equilibrium.

Moment and Center of Mass: One-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $x_1, x_2, \ldots, x_n$.

1. The **moment about the origin** is

$$M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

2. The **center of mass** is

$$\bar{x} = \frac{M_0}{m}$$

where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

EXAMPLE 2 The Center of Mass of a Linear System

Find the center of mass of the linear system shown in Figure 7.55.

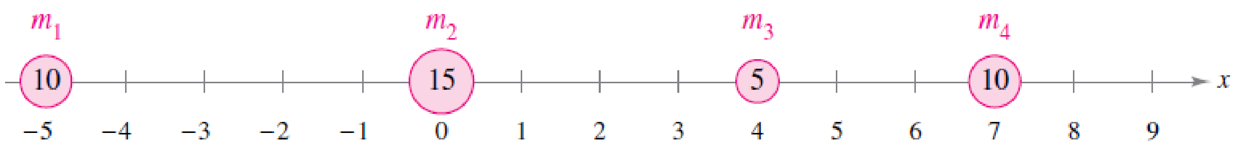


Figure 7.55

Solution The moment about the origin is

$$\begin{aligned} M_0 &= m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 \\ &= 10(-5) + 15(0) + 5(4) + 10(7) \\ &= -50 + 0 + 20 + 70 \\ &= 40. \end{aligned}$$

Moment about origin

Because the total mass of the system is

$$m = 10 + 15 + 5 + 10 = 40$$

Total mass

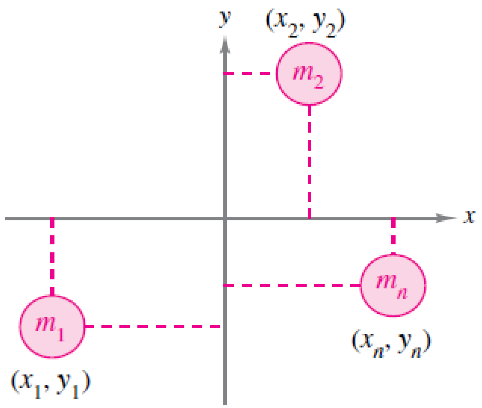
the center of mass is

$$\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1.$$

Center of mass

Note that the point masses will be in equilibrium when the fulcrum is located at $x = 1$.

Center of Mass in a Two-Dimensional System



In a two-dimensional system, there is a moment about the y-axis M_y and a moment about the x-axis M_x.
Figure 7.56

You can extend the concept of moment to two dimensions by considering a system of masses located in the xy-plane at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, as shown in Figure 7.56. Rather than defining a single moment (with respect to the origin), two moments are defined—one with respect to the x-axis and one with respect to the y-axis.

Moments and Center of Mass: Two-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the *y*-axis** is

$$M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

2. The **moment about the *x*-axis** is

$$M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n.$$

3. The **center of mass** $(\overline{x}, \overline{y})$ (or **center of gravity**) is

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}$$

where

$$m = m_1 + m_2 + \cdots + m_n$$

is the **total mass** of the system.

The moment of a system of masses in the plane can be taken about any horizontal or vertical line. In general, the moment about a line is the sum of the product of the masses and the *directed distances* from the points to the line.

$$\text{Moment} = m_1(y_1 - b) + m_2(y_2 - b) + \cdots + m_n(y_n - b) \qquad \text{Horizontal line } y = b$$

$$\text{Moment} = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a) \qquad \text{Vertical line } x = a$$

EXAMPLE 3 The Center of Mass of a Two-Dimensional System

Find the center of mass of a system of point masses $m_1 = 6$, $m_2 = 3$, $m_3 = 2$, and $m_4 = 9$, located at

$$(3, -2), \quad (0, 0), \quad (-5, 3), \quad \text{and} \quad (4, 2)$$

as shown in Figure 7.57.

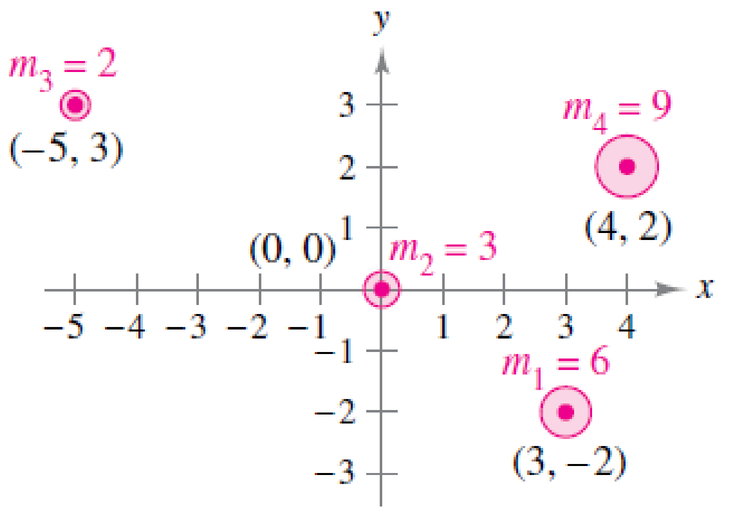


Figure 7.57

Solution

$$\begin{aligned} M &= 6 + 3 + 2 + 9 = 20 && \text{Mass} \\ M_y &= 6(3) + 3(0) + 2(-5) + 9(4) = 44 && \text{Moment about } y\text{-axis} \\ M_x &= 6(-2) + 3(0) + 2(3) + 9(2) = 12 && \text{Moment about } x\text{-axis} \end{aligned}$$

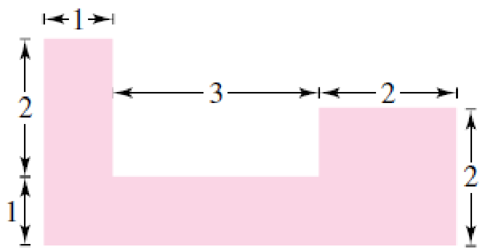
So,

$$\bar{x} = \frac{M_y}{m} = \frac{44}{20} = \frac{11}{5}$$

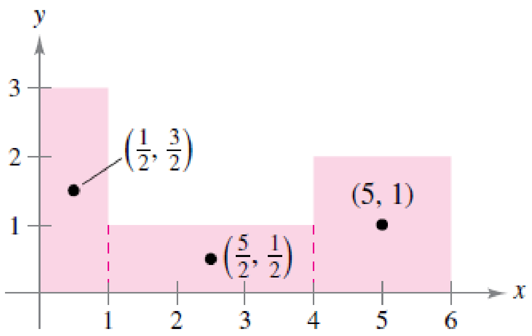
and

$$\bar{y} = \frac{M_x}{m} = \frac{12}{20} = \frac{3}{5}.$$

The center of mass is $\left(\frac{11}{5}, \frac{3}{5}\right)$.



(a) Original region



(b) The centroids of the three rectangles

Figure 7.62

EXAMPLE 6 The Centroid of a Simple Plane Region

Find the centroid of the region shown in Figure 7.62(a).

Solution By superimposing a coordinate system on the region, as shown in Figure 7.62(b), you can locate the centroids of the three rectangles at

$$\left(\frac{1}{2}, \frac{3}{2}\right), \quad \left(\frac{5}{2}, \frac{1}{2}\right), \quad \text{and} \quad (5, 1).$$

Using these three points, you can find the centroid of the region.

$$A = \text{area of region} = 3 + 3 + 4 = 10$$

$$\bar{x} = \frac{(1/2)(3) + (5/2)(3) + (5)(4)}{10} = \frac{29}{10} = 2.9$$

$$\bar{y} = \frac{(3/2)(3) + (1/2)(3) + (1)(4)}{10} = \frac{10}{10} = 1$$

So, the centroid of the region is (2.9, 1). Notice that (2.9, 1) is not the "average" of $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{5}{2}, \frac{1}{2}\right)$, and (5, 1).

THE (FIRST) MOMENT M_L , OF A PLANE AREA with respect to a line ***L*** is the product of the area and the directed distance of its centroid from the line. The moment of a composite area with respect to a line is the sum of the moments of the individual areas with respect to the line.
The moment of a plane area with respect to a coordinate axis may be found as follows:

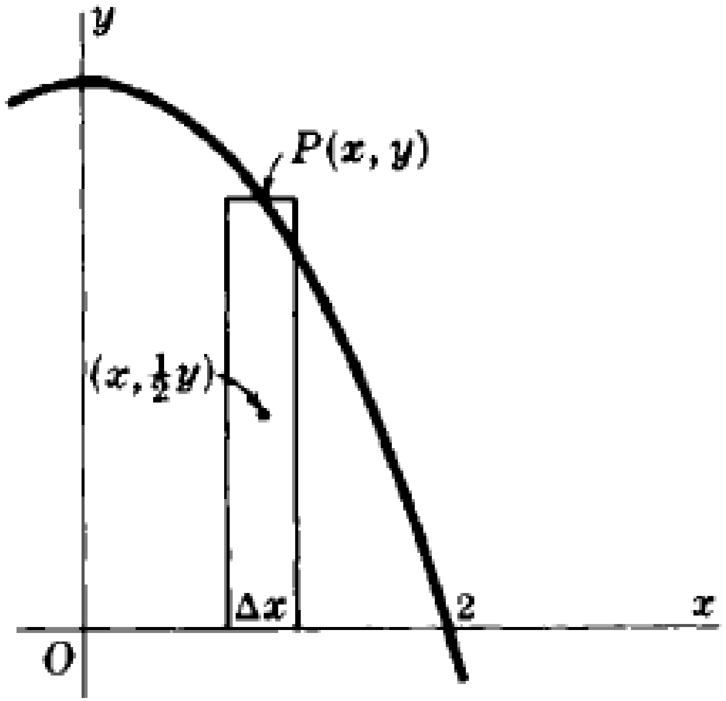
1. Sketch the area, showing a representative strip and the approximating rectangle.
2. Form the product of the area of the rectangle and the distance of its centroid from the axis, and sum for all the rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

For a plane area A having centroid $(\bar{x}, \bar{y})$ and moments M_x and M_y with respect to the x and y axes,

$$A\bar{x} = M_y \qquad \text{and} \qquad A\bar{y} = M_x$$

Example

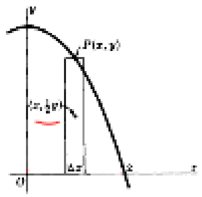
Determine the centroid of the first-quadrant area bounded by the parabola $y = 4 - x^2$.



INTEGRAL CALCULUS

Example

Determine the centroid of the first-quadrant area bounded by the parabola $y = 4 - x^2$.



A, M_x, M_y

$A\bar{x} = M_y$ and $A\bar{y} = M_x$

$\bar{x} = x$

$\bar{y} = \frac{1}{2}y$

$A = \int_0^2 y\,dx$

$= \int_0^2 (4 - x^2)\,dx$

$= 4x - \frac{x^3}{3}\Big|_0^2$

$A = \frac{16}{3}$

$M_x = A \cdot \bar{y}$

$= \int_0^2 y\,dx\,\frac{1}{2}y$

$= \int_0^2 (4 - x^2)\frac{1}{2}(4 - x^2)\,dx$

$= \frac{1}{2}\int_0^2 (4 - x^2)^2\,dx$

$= \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)\,dx$

$= \frac{1}{2}\left| 16x - \frac{8}{3}x^3 + \frac{x^5}{5} \right|_0^2$

$M_x = \frac{128}{15}$

$M_y = A \cdot \bar{x}$

$= \int y\,dx(x)$

$= \int_0^2 xy\,dx$

$= \int_0^2 x(4 - x^2)\,dx$

$= \int_0^2 (4x - x^3)\,dx$

$= 2x^2 - \frac{x^4}{4}\Big|_0^2$

$M_y = 4$

$\bar{x} = \frac{M_y}{A}$

$= \frac{4}{16/3}$

$\bar{x} = \frac{3}{4}$

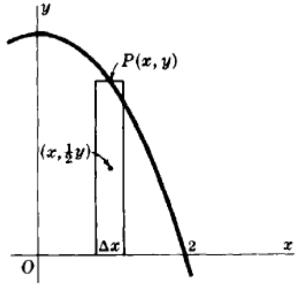
$\bar{y} = \frac{M_x}{A}$

$= \frac{128/15}{16/3}$

$\bar{y} = 8/5$

39

The centroid of the approximating rectangle, shown in Fig. 43-3, is $(x, \frac{1}{2}y)$. Then its area is



$$A = \int_0^2 y\,dx = \int_0^2 (4 - x^2)\,dx = \frac{16}{3}$$

and

$$M_x = \int_0^2 \tfrac{1}{2} y(y\,dx) = \tfrac{1}{2}\int_0^2 (4 - x^2)^2\,dx = \frac{128}{15}$$

$$M_y = \int_0^2 xy\,dx = \int_0^2 x(4 - x^2)\,dx = 4$$

Hence, $\bar{x} = M_y/A = \frac{3}{4}$, $\bar{y} = M_x/A = \frac{8}{5}$, and the centroid has coordinates $(\frac{3}{4}, \frac{8}{5})$.

2. Find the centroid of the first-quadrant area bounded by the parabola $y = x^2$ and the line $y = x$.

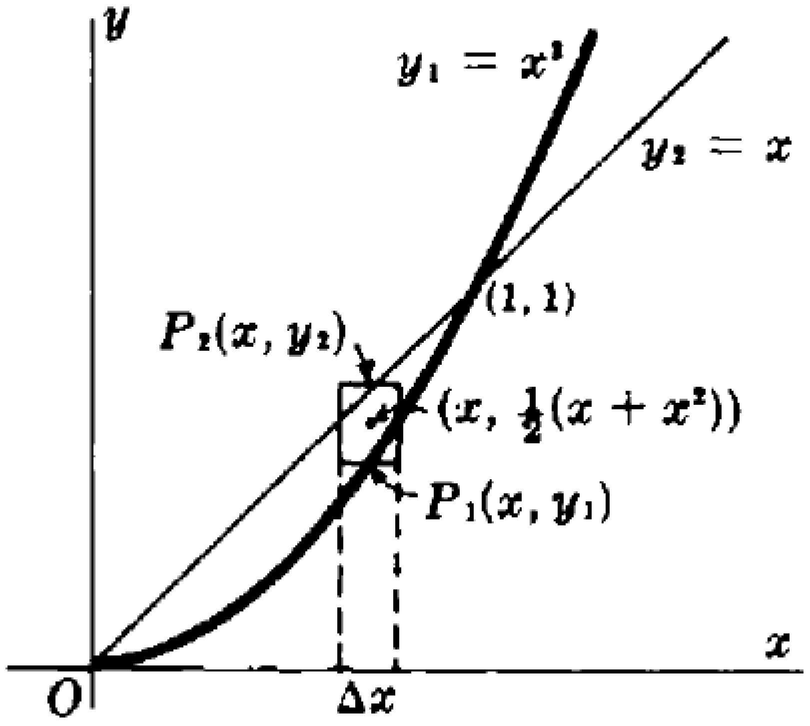


Fig. 43-4

The centroid of the approximating rectangle, shown in Fig. 43-4, is $(x, \frac{1}{2}(x + x^2))$. Then

$$A = \int_0^1 (x - x^2)\, dx = \tfrac{1}{6}$$

$$M_x = \int_0^1 \tfrac{1}{2}(x + x^2)(x - x^2)\, dx = \tfrac{1}{15} \qquad M_y = \int_0^1 x(x - x^2)\, dx = \tfrac{1}{12}$$

Hence, $\bar{x} = M_y/A = \tfrac{1}{2}$, $\bar{y} = M_x/A = \tfrac{2}{5}$, and the coordinates of the centroid are $(\tfrac{1}{2}, \tfrac{2}{5})$.

Find the centroid of the area under the curve $y = 2\sin 3x$ from $x = 0$ to $x = \pi/3$.

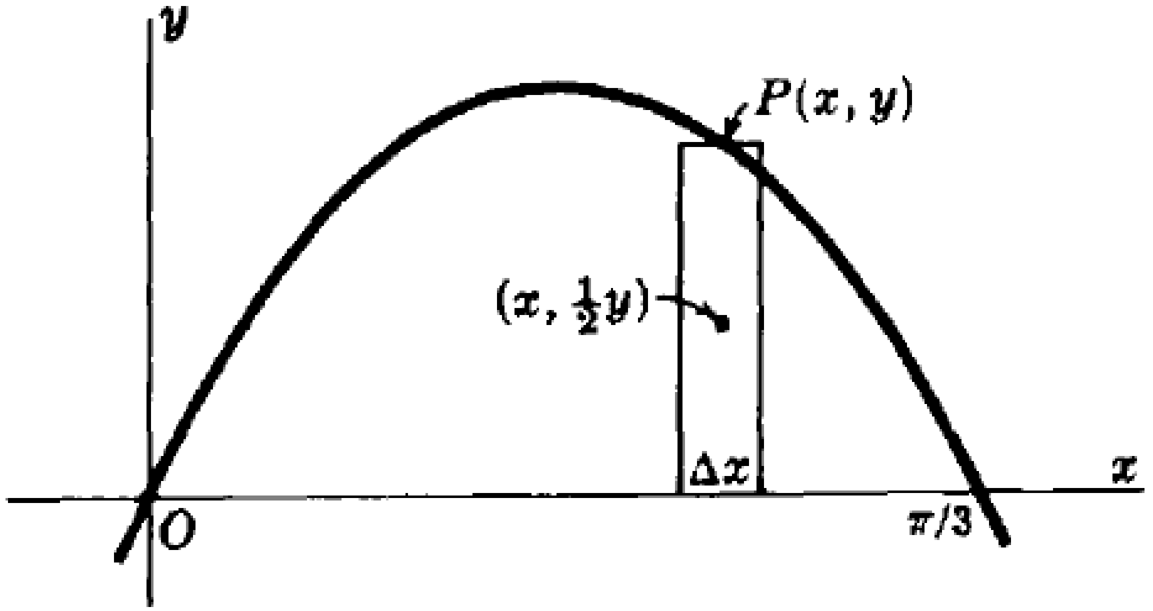


Fig. 43-6

The approximating rectangle, shown in Fig. 43-6, has the centroid $(x, \frac{1}{2}y)$. Then

$$A = \int_0^{\pi/3} y\,dx = \int_0^{\pi/3} 2\sin 3x\,dx = \left[-\frac{2}{3}\cos 3x\right]_0^{\pi/3} = \frac{4}{3}$$

$$M_x = \int_0^{\pi/3} \frac{1}{2}\,y(y\,dx) = 2\int_0^{\pi/3} \sin^2 3x\,dx = 2\left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_0^{\pi/3} = \frac{\pi}{3}$$

$$M_y = \int_0^{\pi/3} xy\,dx = 2\int_0^{\pi/3} x\sin 3x\,dx = \frac{2}{9}\left[\sin 3x - 3x\cos 3x\right]_0^{\pi/3} = \frac{2}{9}\pi$$

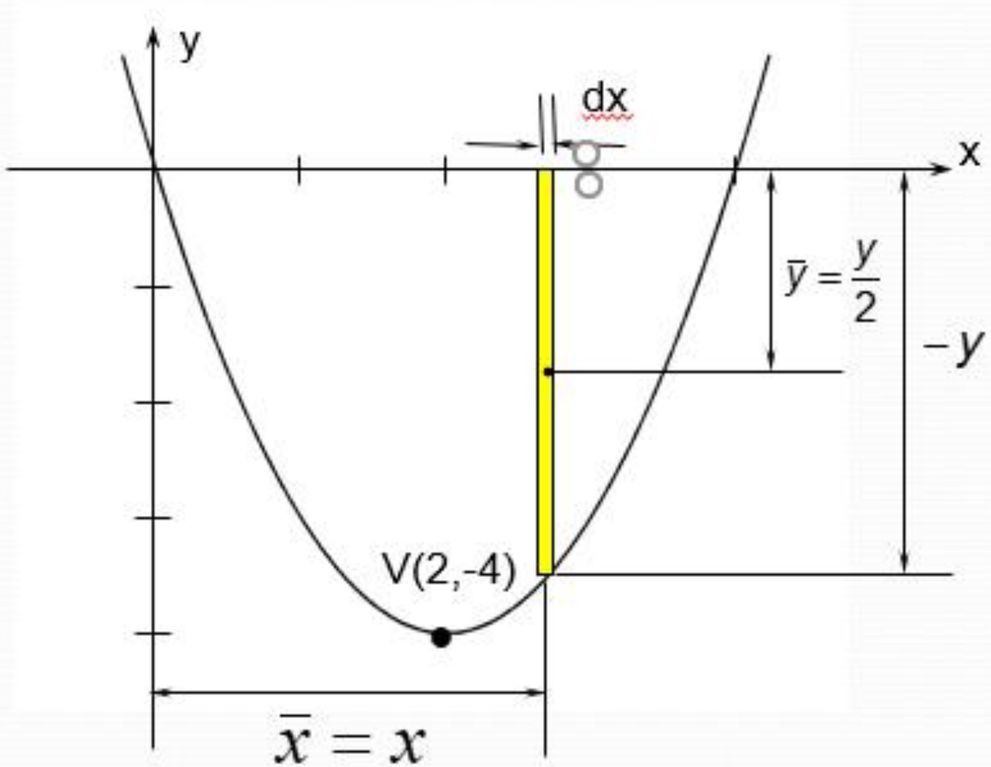
ıe coordinates of the centroid are $(M_y/A, M_x/A) = (\pi/6, \pi/4)$.

INTEGRAL CALCULUS

Example:

Determine the centroid of the fourth-quadrant area bounded by the curve

$y = x^2 - 4x$



Curve: $y = x^2 - 4x$

$$x^2 - 4x + 4 = y + 4$$

$$(x-2)^2 = y + 4$$

$$V(2,-4)$$

$$y = 0;\ x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0;\ x = 4$$

Solving for area A:

$$dA = -\bar{y} \cdot dx$$

$$A = -\int_0^4 y \cdot dx$$

but $y = x^2 - 4x$

$$A = -\int_0^4 \left(x^2 - 4x\right) dx$$

or

$$A = \int_0^4 \left(4x - x^2\right) dx = 4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\Bigg]_0^4$$

$$A = 2(16) - \frac{1}{3}(64) = 32 - \frac{64}{3}$$

$$A = \frac{96 - 64}{3} = \frac{32}{3} \text{ sq. units}$$

INTEGRAL CALCULUS

$$M_x = A \cdot \bar{y}$$

$$M_x = -\int_0^4 y \cdot \bar{y} \cdot dx$$

$$\text{but } \bar{y} = \frac{y}{2}$$

$$M_x = -\int_0^4 y \cdot \frac{y}{2} \cdot dx$$

$$= \frac{1}{2}\int_0^2 (4 - x^2) y \cdot dx$$

$$\text{since } y = 4 - x^2$$

$$M_x = -\frac{1}{2}\int_0^4 y^2 \cdot dx$$

$$M_x = -\frac{1}{2}\int_0^4 (x^2 - 4x)^2 dx$$

$$M_x = -\frac{1}{2}\int_0^4 (x^4 - 8x^3 + 16x^2) dx$$

$$M_x = -\frac{1}{2}\left[\frac{x^5}{5} - 8 \cdot \frac{x^4}{4} + 16 \cdot \frac{x^3}{3}\right]_0^4$$

$$M_x = -\frac{1}{2}\left[\frac{1}{5}(1024) - 2(256) + \frac{16}{3}(64)\right]$$

$$M_x = -\frac{1}{2}\left[\frac{3072 - 7680 + 5120}{15}\right]$$

$$M_x = -\frac{1}{2}\left[\frac{512}{15}\right] = -\frac{256}{15} \text{ cu. units}$$

$$M_y = A \cdot \bar{x}$$

$$M_y = \int_0^4 (4x - x^2) \bar{x} \cdot dx$$

$$\text{but } \bar{x} = x$$

$$M_y = \int_0^4 (4x - x^2) x dx$$

$$M_y = \int_0^4 (4x^2 - x^3) dx$$

$$M_y = \left[4 \cdot \frac{x^3}{3} - \frac{x^4}{4}\right]_0^4$$

$$M_y = \left[\frac{4}{3}(64) - 64\right]$$

$$M_y = \left[\frac{256}{3} - 64\right]$$

$$M_y = \left[\frac{256 - 192}{3}\right]$$

$$M_y = \frac{64}{3} \text{ cu. units}$$

INTEGRAL CALCULUS

$$M_x = A\bar{y}$$

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{-256}{15}}{\frac{32}{3}}$$

$$\bar{y} = \frac{-8}{5}\,units$$

$$M_y = A \cdot \bar{x}$$

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{64}{3}}{\frac{32}{3}}$$

$$\bar{x} = 2 \textbf{ units}$$

$$\therefore C\left(2, \frac{-8}{5}\right)$$