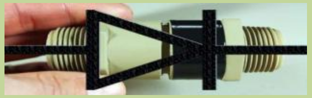
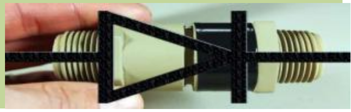
CALCULUS 2

AREAS AND VOLUME/ WORK , MULTIPLE INTEGRALS

PRESENTED BY ENGR. JOHN R. REJANO,ECE



OBJECTIVES



- find the area between a curve y = f(x) and an interval on the x-axis.
- find the area between two curves
- Learn solid of revolution, method of washers and disks
- Learn centroid, application of integrals
- Learn double and triple integrals

SOME PROBLEMS INVOLVING INTEGRAL

AVERAGE VELOCITY REVISITED

Let $s = s(t)$ denote the position function of a particle in rectilinear motion. In Section 2.1 we defined the average velocity v_{ave} of the particle over the time interval $[t_0, t_1]$ to be

$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Let $v(t) = s'(t)$ denote the velocity function of the particle. We saw in Section 5.7 that integrating $s'(t)$ over a time interval gives the displacement of the particle over that interval. Thus,

$$\int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0)$$

It follows that

$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} v(t)\,dt \tag{1}$$

▶ **Example 1** Suppose that a particle moves along a coordinate line so that its velocity at time t is $v(t) = 2 + \cos t$. Find the average velocity of the particle during the time interval $0 \leq t \leq \pi$.

Solution. From (1) the average velocity is

$$\frac{1}{\pi - 0} \int_0^{\pi} (2 + \cos t)\, dt = \frac{1}{\pi} \left[2t + \sin t\right]_0^{\pi} = \frac{1}{\pi}(2\pi) = 2$$ ◀

AVERAGE VALUE OF A CONTINUOUS FUNCTION

In scientific work, numerical information is often summarized by an *average value* or *mean value* of the observed data. There are various kinds of averages, but the most common is the ***arithmetic mean*** or ***arithmetic average***, which is formed by adding the data and dividing by the number of data points. Thus, the arithmetic average $\overline{a}$ of n numbers $a_1, a_2, \ldots, a_n$ is

$$\overline{a} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \frac{1}{n}\sum_{k=1}^{n} a_k$$

In the case where the a_k's are values of a function f, say,

$$a_1 = f(x_1), a_2 = f(x_2), \ldots, a_n = f(x_n)$$

then the arithmetic average $\overline{a}$ of these function values is

$$\overline{a} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

We will now show how to extend this concept so that we can compute not only the arithmetic average of finitely many function values but an average of *all* values of $f(x)$ as x varies over a closed interval $[a, b]$. For this purpose recall the Mean-Value Theorem for Integrals (5.6.2), which states that if f is continuous on the interval $[a, b]$, then there is at least one point x^* in this interval such that

$$\int_a^b f(x)\,dx = f(x^*)(b - a)$$

The quantity

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

will be our candidate for the average value of f over the interval $[a, b]$. To explain what motivates this, divide the interval $[a, b]$ into n subintervals of equal length

$$\Delta x = \frac{b-a}{n} \tag{2}$$

and choose arbitrary points $x_1^*, x_2^*, \ldots, x_n^*$ in successive subintervals. Then the arithmetic average of the values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\text{ave} = \frac{1}{n}[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

or from (2)

$$\text{ave} = \frac{1}{b-a}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Taking the limit as $n \to +\infty$ yields

$$\lim_{n\to+\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since this equation describes what happens when we compute the average of "more and more" values of $f(x)$, we are led to the following definition.

Note that the Mean-Value Theorem for Integrals, when expressed in form (3), ensures that there is always at least one point x^* in $[a, b]$ at which the value of f is equal to the average value of f over the interval.

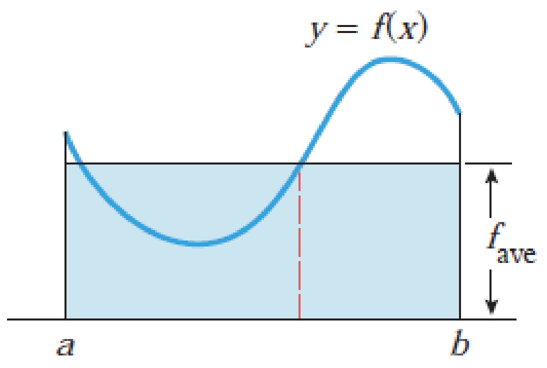
5.8.1 DEFINITION If f is continuous on $[a, b]$, then the ***average value*** (or ***mean value***) of f on $[a, b]$ is defined to be

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{3}$$

REMARK When f is nonnegative on $[a, b]$, the quantity f_{ave} has a simple geometric interpretation, which can be seen by writing (3) as

$$f_{\text{ave}} \cdot (b-a) = \int_a^b f(x)\,dx$$

The left side of this equation is the area of a rectangle with a height of f_{ave} and base of length $b - a$, and the right side is the area under $y = f(x)$ over $[a, b]$. Thus, f_{ave} is the height of a rectangle constructed over the interval $[a, b]$, whose area is the same as the area under the graph of f over that interval (Figure 5.8.1).



▲ Figure 5.8.1

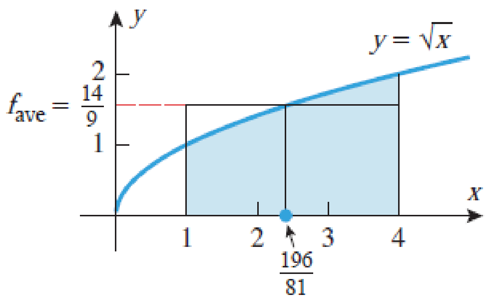
▶ Example 2 Find the average value of the function $f(x) = \sqrt{x}$ over the interval $[1, 4]$, and find all points in the interval at which the value of f is the same as the average.

Solution.

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2x^{3/2}}{3}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{14}{9} \approx 1.6$$

The x-values at which $f(x) = \sqrt{x}$ is the same as this average satisfy $\sqrt{x} = 14/9$, from which we obtain $x = 196/81 \approx 2.4$ (Figure 5.8.2). ◀

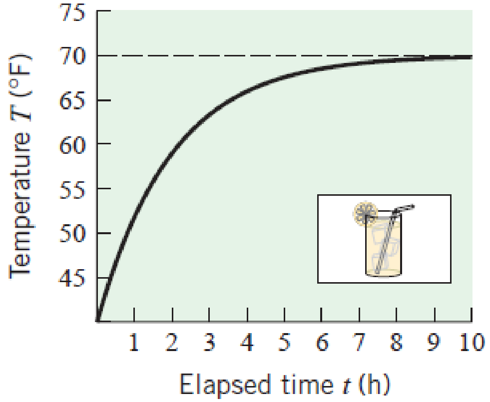


▲ Figure 5.8.2

▶ **Example 3** A glass of lemonade with a temperature of 40°F is left to sit in a room whose temperature is a constant 70°F. Using a principle of physics called ***Newton's Law of Cooling***, one can show that if the temperature of the lemonade reaches 52°F in 1 hour, then the temperature T of the lemonade as a function of the elapsed time t is modeled by the equation

$$T = 70 - 30e^{-0.5t}$$

where T is in degrees Fahrenheit and t is in hours. The graph of this equation, shown in Figure 5.8.3, conforms to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room. Find the average temperature T_{ave} of the lemonade over the first 5 hours.



▲ **Figure 5.8.3**

Solution. From Definition 5.8.1 the average value of T over the time interval [0, 5] is

$$T_{\text{ave}} = \frac{1}{5}\int_0^5 (70 - 30e^{-0.5t})\,dt \tag{4}$$

To evaluate the definite integral, we first find the indefinite integral

$$\int (70 - 30e^{-0.5t})\,dt$$

by making the substitution

$$u = -0.5t \quad \text{so that} \quad du = -0.5\,dt \quad (\text{or } dt = -2\,du)$$

Thus,

$$\int (70 - 30e^{-0.5t})\,dt = \int (70 - 30e^{u})(-2)\,du = -2(70u - 30e^{u}) + C$$

$$= -2[70(-0.5t) - 30e^{-0.5t}] + C = 70t + 60e^{-0.5t} + C$$

and (4) can be expressed as

$$T_{\text{ave}} = \frac{1}{5}\left[70t + 60e^{-0.5t}\right]_0^5 = \frac{1}{5}\left[(350 + 60e^{-2.5}) - 60\right]$$

$$= 58 + 12e^{-2.5} \approx 59°\text{F}$$ ◀

AVERAGE VALUE AND AVERAGE VELOCITY

We now have two ways to calculate the average velocity of a particle in rectilinear motion, since

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} v(t)\, dt \tag{5}$$

and both of these expressions are equal to the average velocity. The left side of (5) gives the average rate of change of s over $[t_0, t_1]$, while the right side gives the average value of $v = s'$ over the interval $[t_0, t_1]$. That is, *the average velocity of the particle over the time interval* $[t_0, t_1]$ *is the same as the average value of the velocity function over that interval.*

Since velocity functions are generally continuous, it follows from the marginal note associated with Definition 5.8.1 that a particle's average velocity over a time interval matches the particle's velocity at some time in the interval.

▶ **Example 4** Show that if a body released from rest (initial velocity zero) is in free fall, then its average velocity over a time interval $[0, T]$ during its fall is its velocity at time $t = T/2$.

Solution. It follows from Formula (16) of Section 5.7 with $v_0 = 0$ that the velocity function of the body is $v(t) = -gt$. Thus, its average velocity over a time interval $[0, T]$ is

$$\begin{aligned}
v_{\text{ave}} &= \frac{1}{T - 0} \int_0^T v(t)\, dt \\
&= \frac{1}{T} \int_0^T -gt\, dt \\
&= -\frac{g}{T} \left[\frac{1}{2}t^2\right]_0^T = -g \cdot \frac{T}{2} = v\left(\frac{T}{2}\right)
\end{aligned}$$ ◀

Area of a Region Between Two Curves

With a few modifications, you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. Also, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, as shown in Figure 7.1. You can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 7.2.

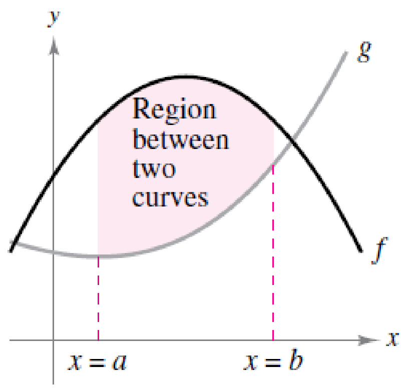
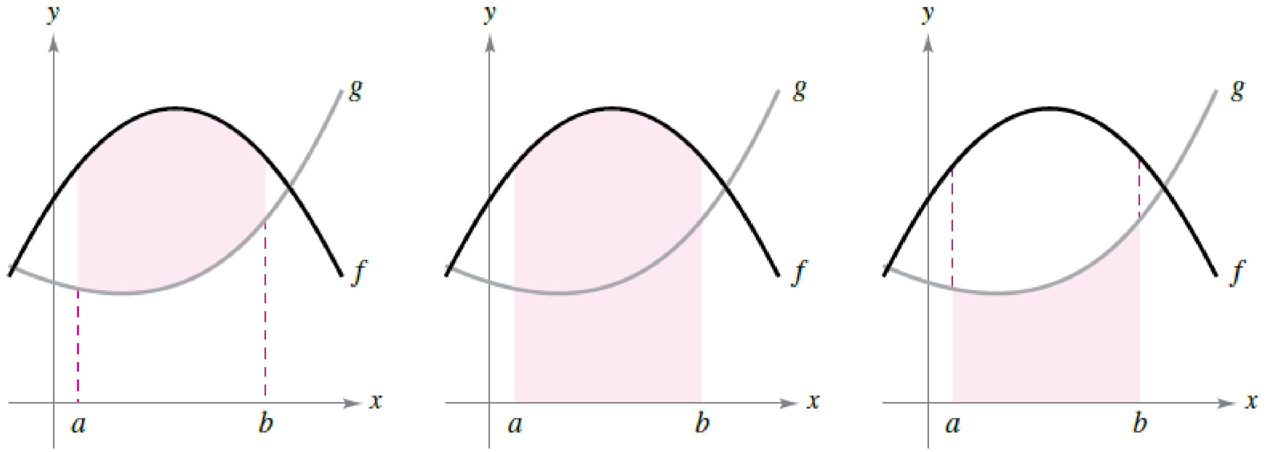


Figure 7.1



Area of region between f and g	=	Area of region under f	−	Area of region under g

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 7.2

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 7.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

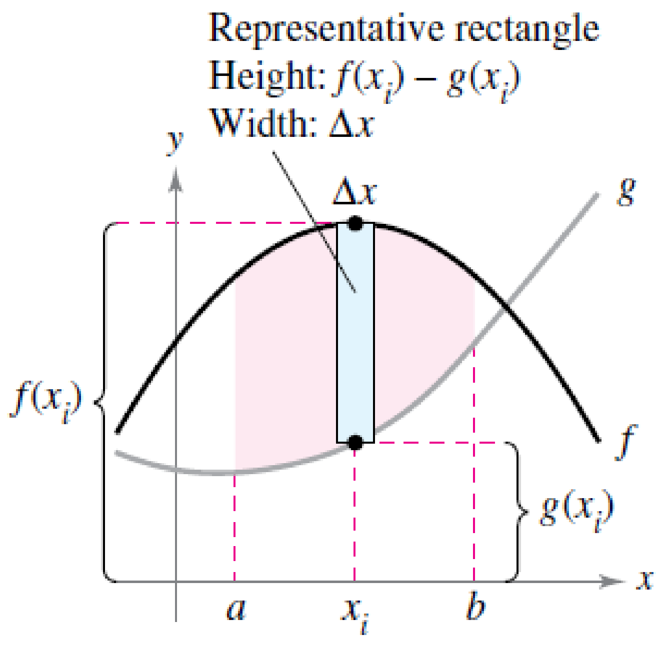


Figure 7.3

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the region is

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x \\ &= \int_a^b [f(x) - g(x)]\,dx.\end{aligned}$$

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE 1 Finding the Area of a Region Between Two Curves

Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \le f(x)$ for all x in $[0, 1]$, as shown in Figure 7.5. So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\Delta x$$
$$= [(x^2 + 2) - (-x)]\Delta x$$

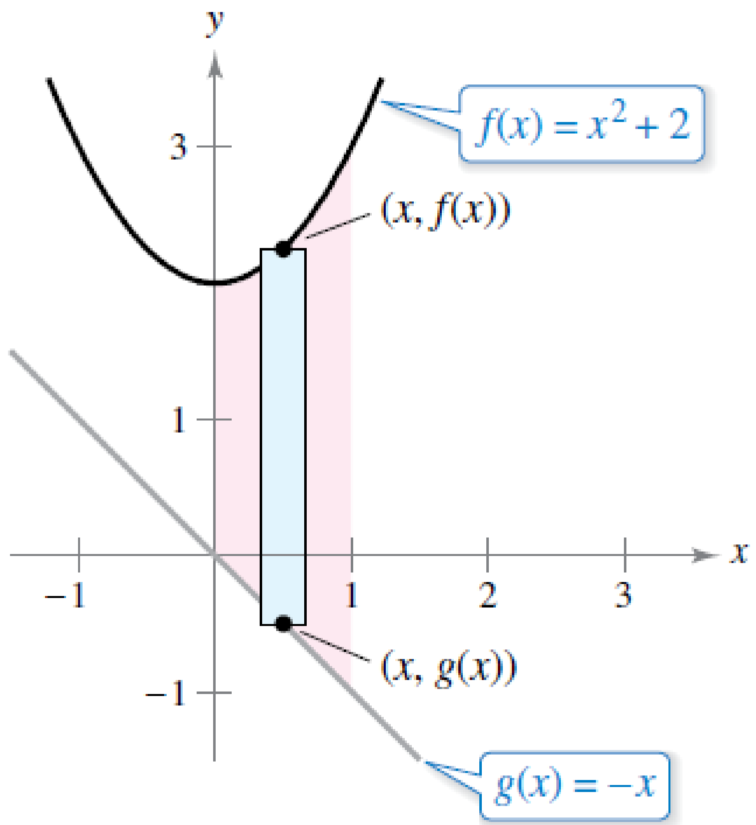
and the area of the region is

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \le f(x)$ for all x in $[0, 1]$, as shown in Figure 7.5. So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\Delta x$$
$$= [(x^2 + 2) - (-x)]\Delta x$$

and the area of the region is

$$A = \int_a^b [f(x) - g(x)]\,dx$$
$$= \int_0^1 [(x^2 + 2) - (-x)]\,dx$$
$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1$$
$$= \frac{1}{3} + \frac{1}{2} + 2$$
$$= \frac{17}{6}.$$



Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$
Figure 7.5

Area of a Region Between Intersecting Curves

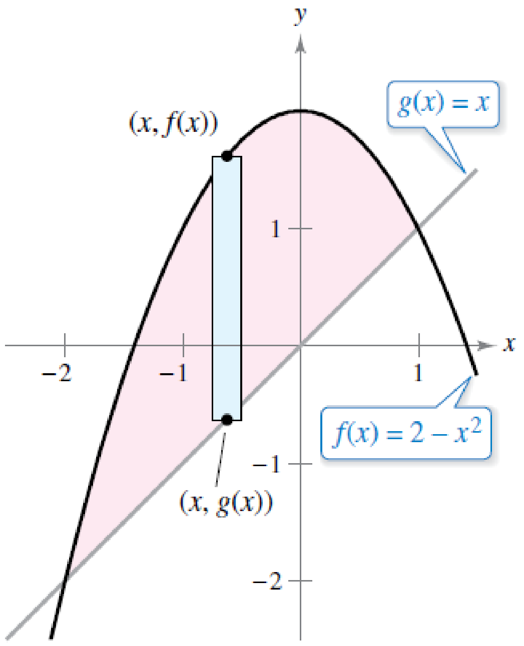
In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly. A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of a and b must be calculated.

EXAMPLE 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution In Figure 7.6, notice that the graphs of f and g have two points of intersection. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$2 - x^2 = x \quad \text{Set } f(x) \text{ equal to } g(x).$$
$$-x^2 - x + 2 = 0 \quad \text{Write in general form.}$$
$$-(x + 2)(x - 1) = 0 \quad \text{Factor.}$$
$$x = -2 \text{ or } 1 \quad \text{Solve for } x.$$



Region bounded by the graph of f and the graph of g
Figure 7.6

So, $a = -2$ and $b = 1$. Because $g(x) \le f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

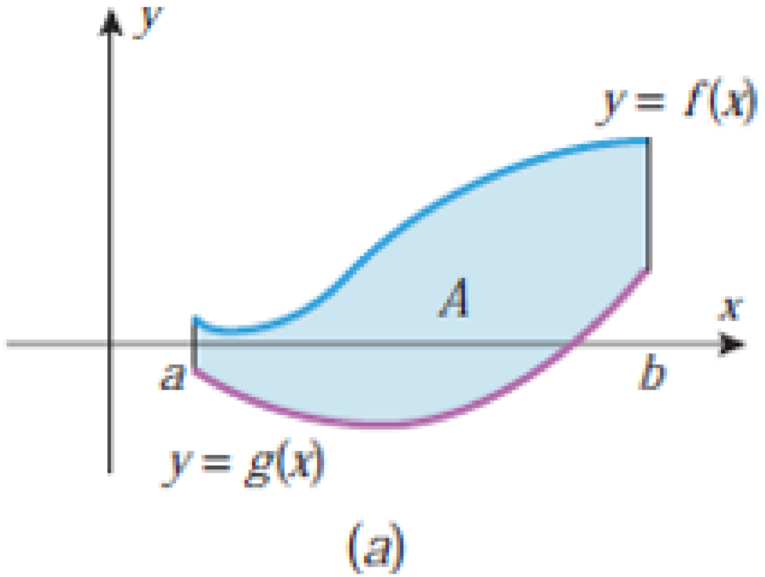
$$\Delta A = [f(x) - g(x)]\Delta x = [(2 - x^2) - x]\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\, dx$$
$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$
$$= \frac{9}{2}.$$

6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

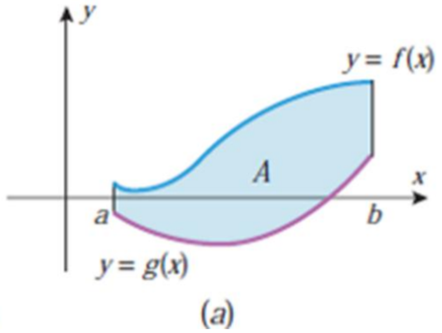
$$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$



▶ **Figure 6.1.3**

SUMMARY OF FORMULAS:
AREAS

$$A = \int_a^b [f(x) - g(x)]\, dx$$



▶ Figure 6.1.3

$$A = \int_c^d [w(y) - v(y)]\, dy$$

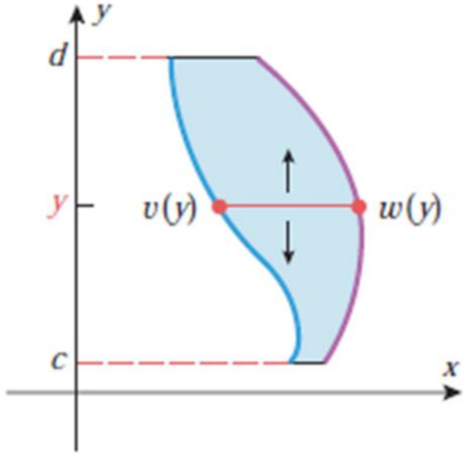
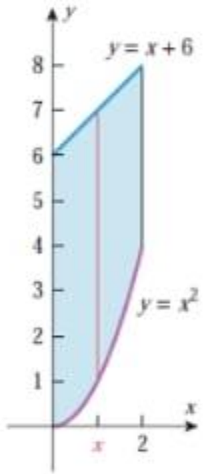


Figure 6.1.12

► **Example 1** Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.



▲ Figure 6.1.4

$$A = \int_a^b f(x) - g(x)\,dx$$

$$f(x) = x+6 \quad g(x) = x^2 \quad a = 0 \quad b = 2$$

$$A = \int_0^2 (x+6) - (x^2)\,dx$$

$$= \int_0^2 (x + 6 - x^2)\,dx$$

$$= \frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_0^2$$

$$= \left[\frac{2^2}{2} + 6(2) - \frac{2^3}{3}\right] - \left[\frac{0^2}{2} + 6(0) - \frac{0^3}{3}\right]$$

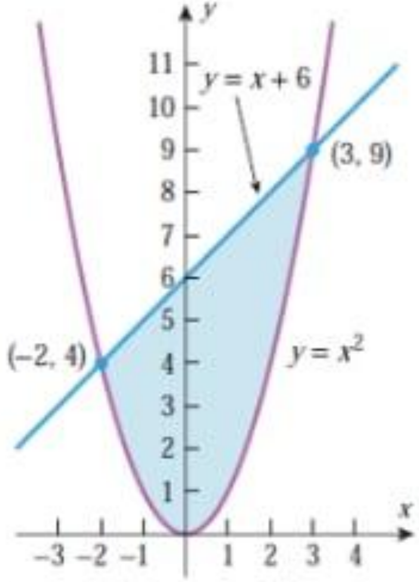
$$= \left[2 + 12 - \frac{8}{3}\right] - (0]$$

$$A = \frac{34}{3}$$

▸ **Example 2** Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3 \quad x = -2$$

$$y = x + 6$$
$$x = 3$$
$$y = 9$$
$$x = -2$$
$$y = 4$$



▲ **Figure 6.1.6**

$$A = \int_a^b f(x) - g(x)\,dx \qquad f(x) = x + 6 \qquad g(x) = x^2$$

$$= \int_{-2}^{3} (x+6) - (x^2)\,dx$$

$$= \int_{-2}^{3} (x + 6 - x^2)\,dx$$

$$= \frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_{-2}^{3}$$

$$= \left[\frac{3^2}{2} + 6(3) - \frac{3^3}{3}\right] - \left[\frac{-2^2}{2} + 6(-2) - \frac{(-2)^3}{3}\right]$$

$$= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right]$$

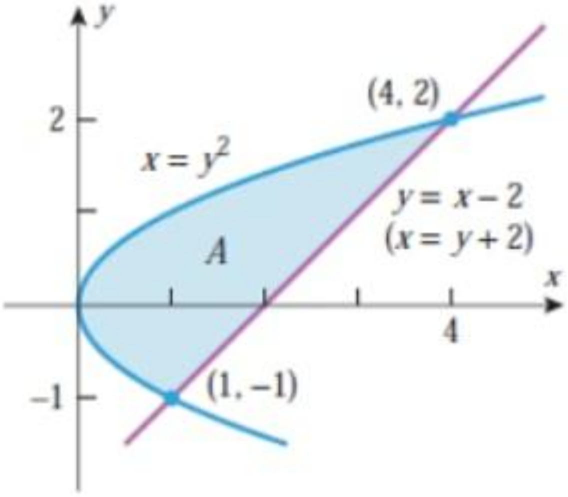
$$= \frac{27}{2} + \frac{22}{3} = \boxed{\frac{125}{6}}$$

Ex. 3 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

$x = y^2$ $\quad$ $y = x - 2$ $\quad$ $w(y) = y + 2$

$y + 2 = x$ $\quad$ $v(y) = y^2$

$x = y + 2$



(a)

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2 \quad y = -1$

if $y = 2$: $x = y + 2 = 2 + 2$, $x = 4$, $(4, 2)$

if $y = -1$: $x = -1 + 2$, $x = 1$, $(1, -1)$

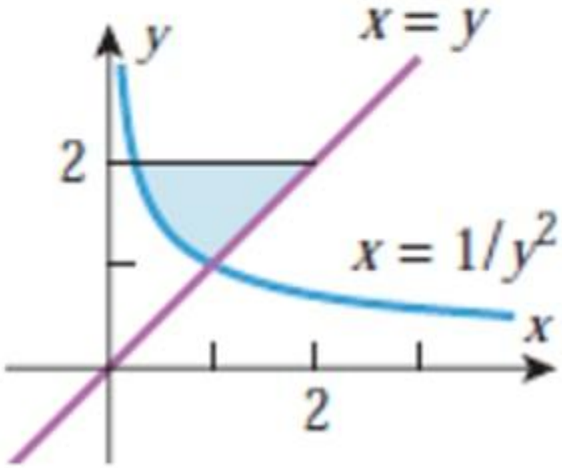
$$A = \int_c^d w(y) - v(y)\,dy = \int_{-1}^{2} (y+2) - y^2\,dy = \int_{-1}^{2} (y + 2 - y^2)\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_{-1}^{2}$$

$$= \left[2 + 4 - \frac{8}{3}\right] - \left[\frac{1}{2} - 2 + \frac{1}{3}\right]$$

$$\boxed{= \frac{9}{2}}$$

4) find the area of shaded region



$w(y) = y$

$v(y) = \frac{1}{y^2}$

$A = \int_c^d w(y) - v(y)\,dy$

$= \int_1^2 (y - \frac{1}{y^2})\,dy$

$= \frac{y^2}{2} + \frac{1}{y} \Big|_1^2$

$= \left(\frac{2^2}{2} + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right)$

$= \frac{5}{2} - \frac{3}{2}$

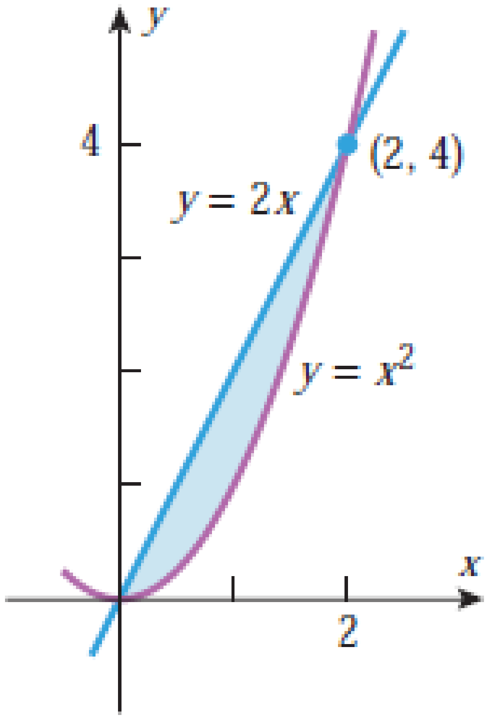
$= \frac{2}{2}$ or 1

INTEGRAL CALCULUS

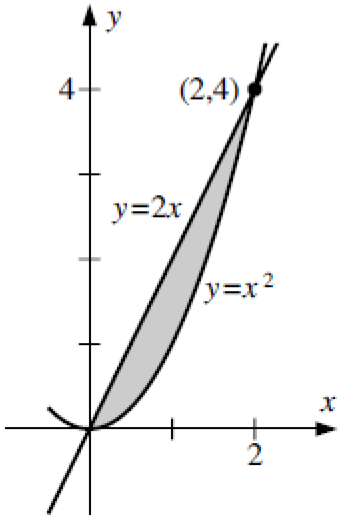
Exercises

Find the area of the shaded region by (a) integrating with respect to *x* and (b) integrating with respect to *y*.

5.



5. (a) $A = \int_0^2 (2x - x^2)\,dx = 4/3.$ (b) $A = \int_0^4 (\sqrt{y} - y/2)\,dy =$

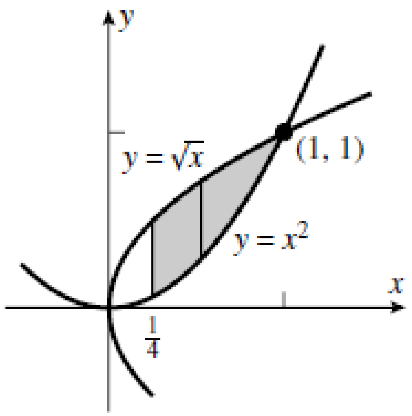


Exercises

Sketch the region enclosed by the curves and find its area.

7. $y = x^2,\ y = \sqrt{x},\ x = \frac{1}{4},\ x = 1$

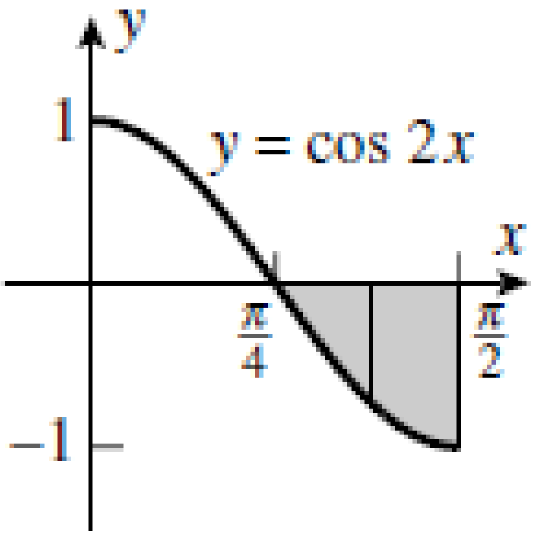
7. $A = \int_{1/4}^{1} (\sqrt{x} - x^2)\, dx = 49/192.$



Exercises

Sketch the region enclosed by the curves and find its area.

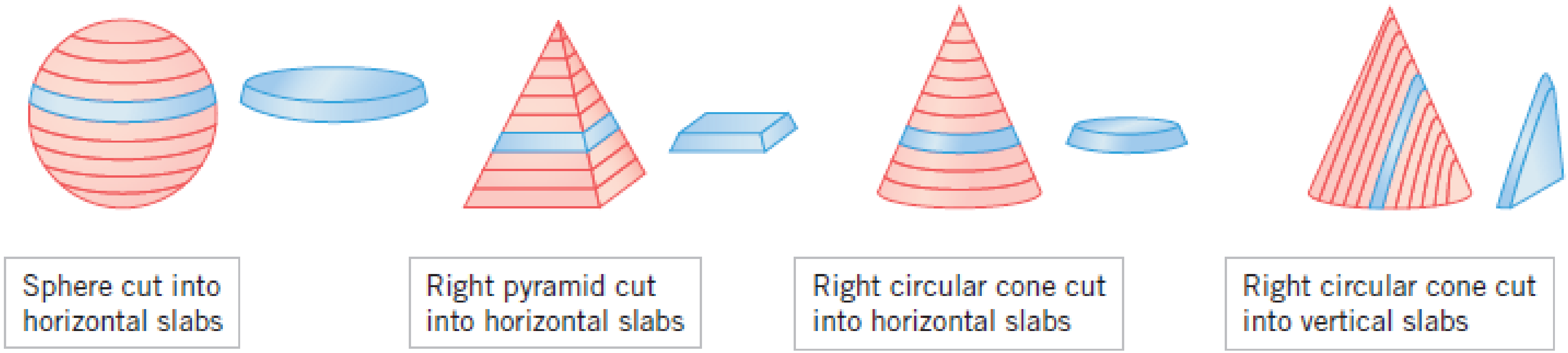
9. $y = \cos 2x, \; y = 0, \; x = \pi/4, \; x = \pi/2$



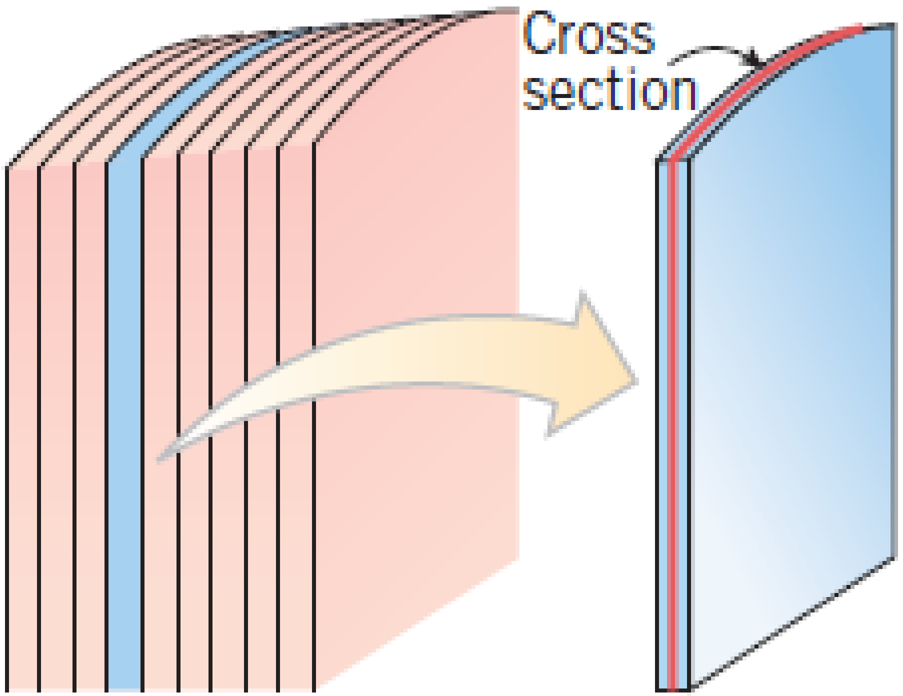
9. $A = \int_{\pi/4}^{\pi/2} (0 - \cos 2x)\, dx = -\int_{\pi/4}^{\pi/2} \cos 2x\, dx = 1/2.$

VOLUMES BY SLICING; DISKS AND WASHERS

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).



▲ Figure 6.2.1



In a thin slab, the cross sections do not vary much in size and shape.

▲ **Figure 6.2.2**

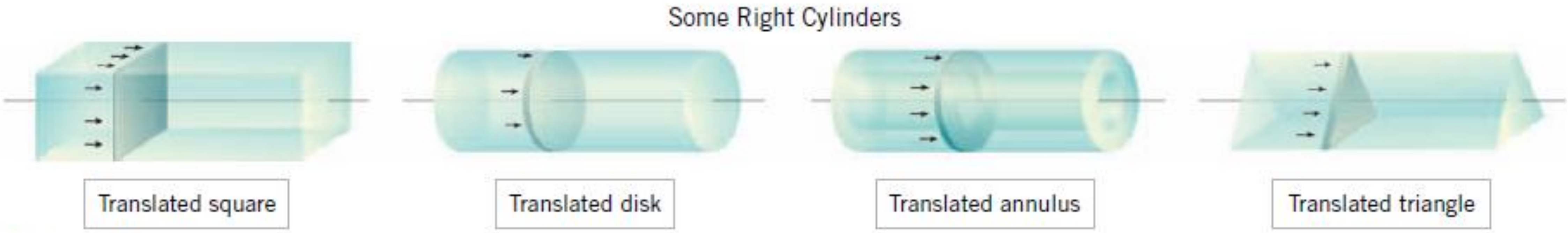
What makes this method work is the fact that a *thin* slab has a cross section that does not In a thin slab, the cross sections do not vary much in size and shape.
(Figure 6.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set up a Riemann sum whose limit is the volume of the entire solid

One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius r, since all cross sections taken perpendicular to the central axis are circular regions of radius r. The volume V of a right circular cylinder of radius r and height h can be expressed in terms of the height and the area of a cross section as

$$V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}] \qquad (1)$$

INTEGRAL CALCULUS

- This is a special case of a more general volume formula that applies to solids called right cylinders.
- A ***right cylinder*** is a solid that is generated when a plane region is translated along a line or ***axis*** that is perpendicular to the region (Figure 6.2.3).

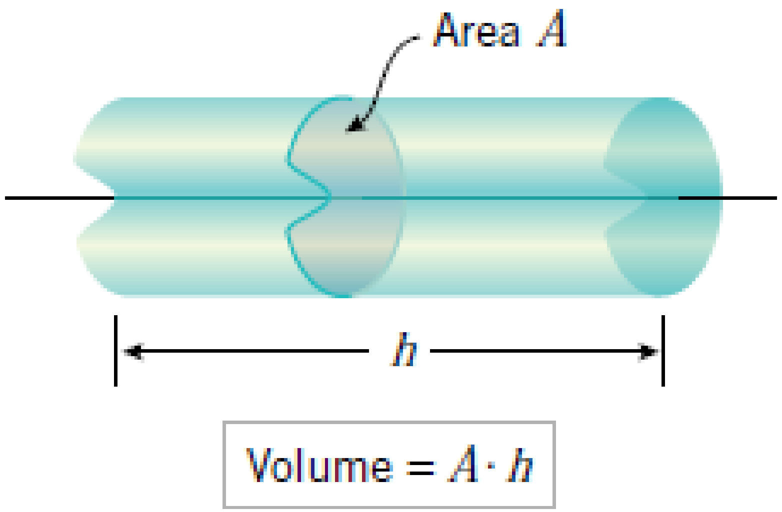


▲ Figure 6.2.3

If a right cylinder is generated by translating a region of area A through a distance h, then h is called the ***height*** (or sometimes the ***width***) of the cylinder, and the volume V of the cylinder is defined to be

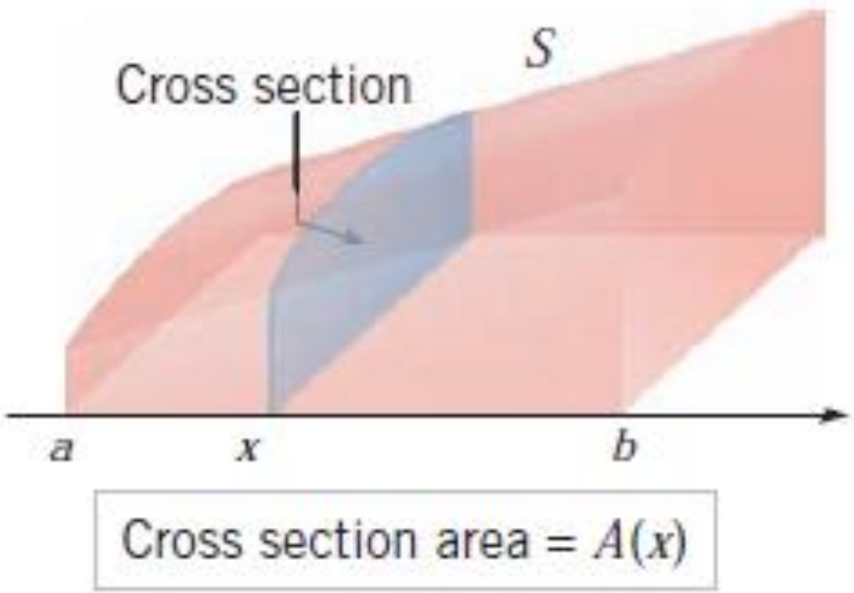
$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \tag{2}$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right *circular* cylinder.



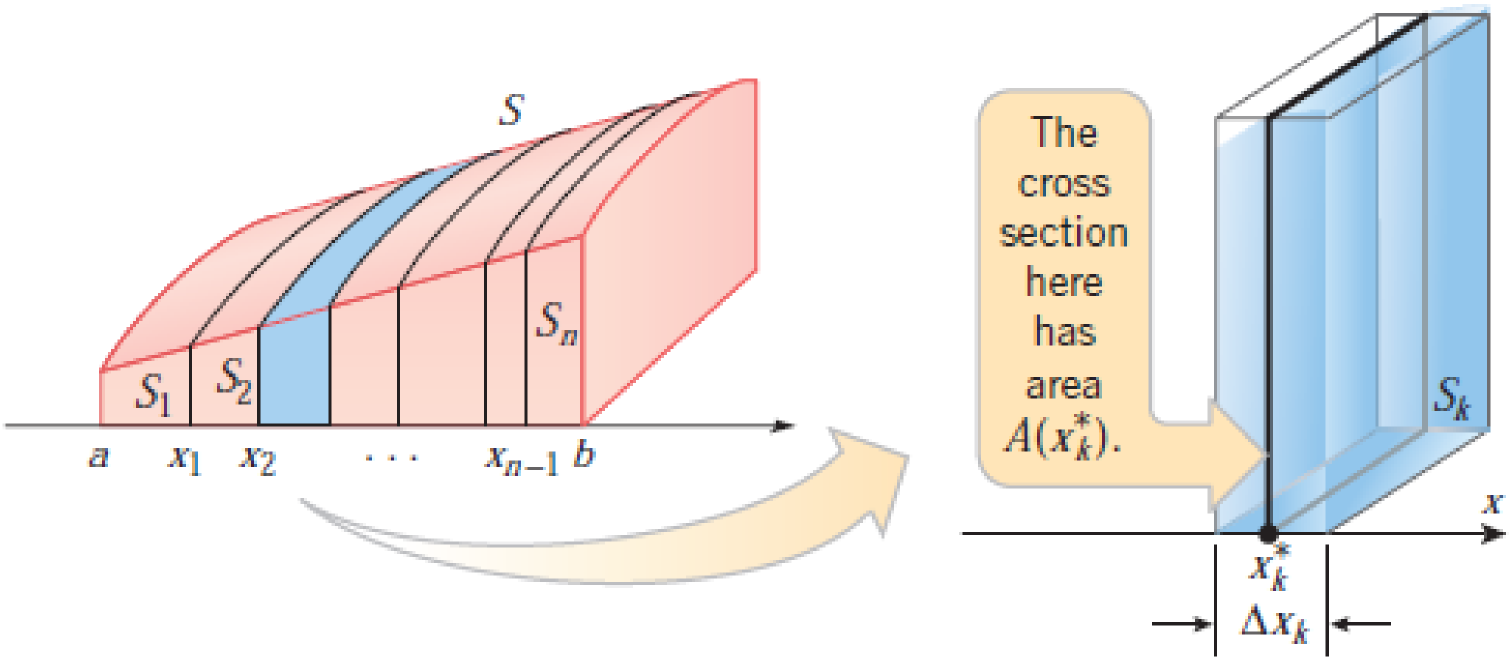
▲ **Figure 6.2.4**

6.2.1 PROBLEM Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at $x = a$ and $x = b$ (Figure 6.2.5). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.



▲ **Figure 6.2.5**

To solve this problem we begin by dividing the interval $[a, b]$ into n subintervals, thereby dividing the solid into n slabs as shown in the left part of Figure 6.2.6. If we assume that the width of the kth subinterval is Δx_k, then the volume of the kth slab can be approximated by the volume $A(x_k^*)\Delta x_k$ of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is a point in the kth subinterval (see the right part of Figure 6.2.6).



▶ **Figure 6.2.6**

Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

INTEGRAL CALCULUS

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

There is a similar result for cross sections perpendicular to the y-axis.

6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

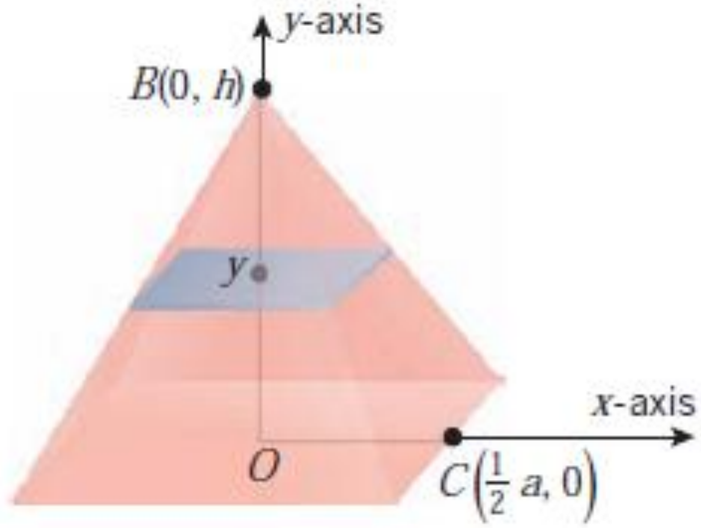
$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

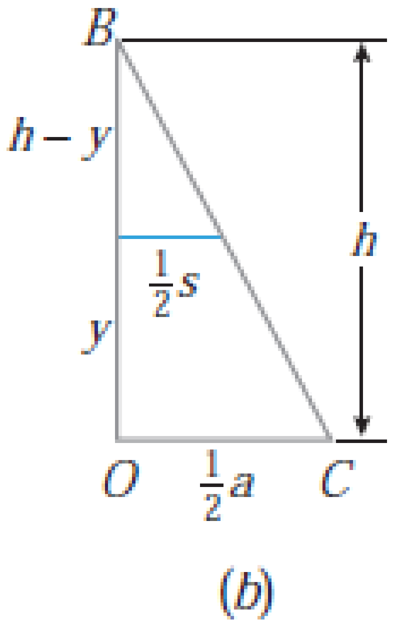
In words, these formulas state: ***The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.***

▸ **Example 1** Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution. As illustrated in Figure 6.2.7a, we introduce a rectangular coordinate system in which the y-axis passes through the apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.



(a)



▲ **Figure 6.2.7**

At any y in the interval $[0, h]$ on the y-axis, the cross section perpendicular to the y-axis is a square. If s denotes the length of a side of this square, then by similar triangles (Figure 6.2.7b)

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h}(h-y)$$

Thus, the area $A(y)$ of the cross section at y is

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$

and by (4) the volume is

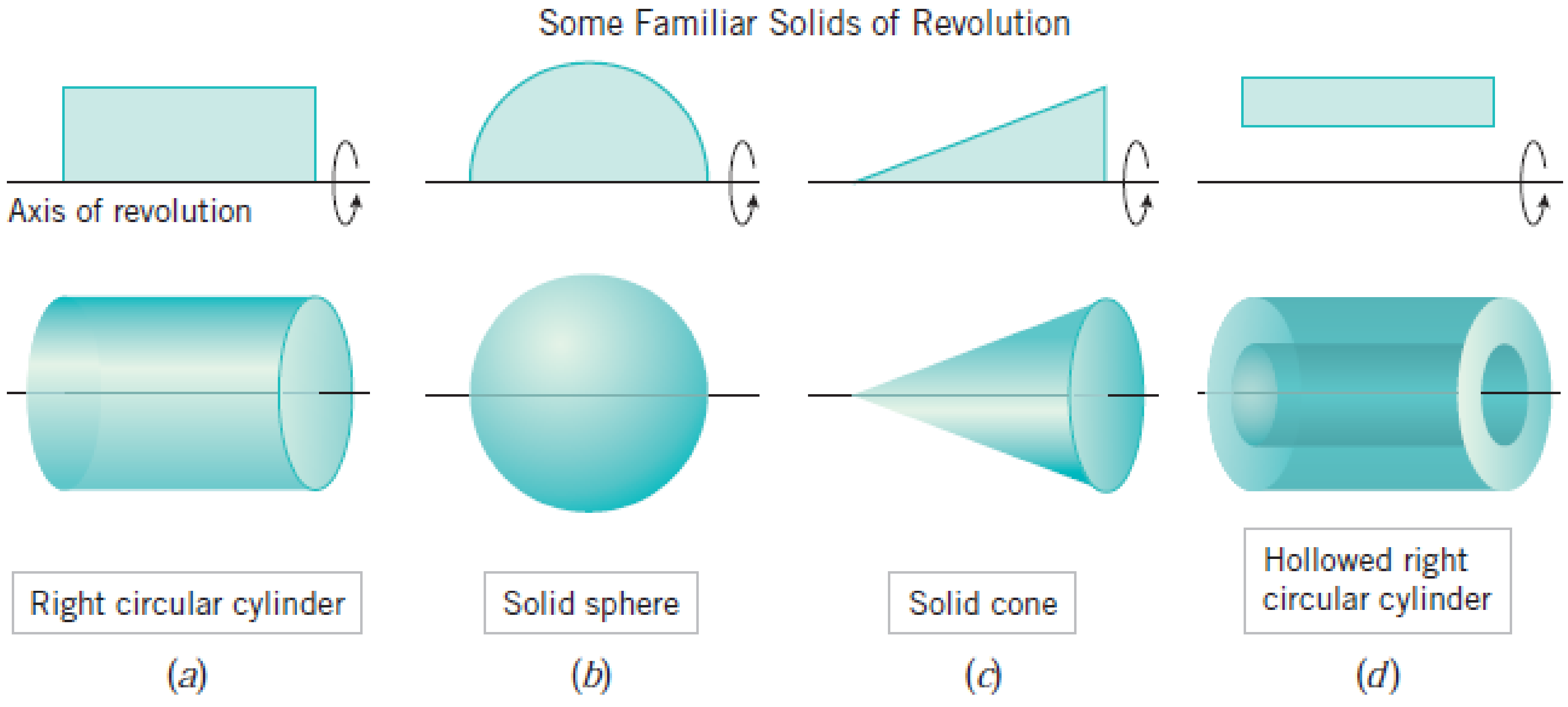
$$V = \int_0^h A(y)\,dy = \int_0^h \frac{a^2}{h^2}(h-y)^2\,dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy$$

$$= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_{y=0}^{h} = \frac{a^2}{h^2}\left[0 + \frac{1}{3}h^3\right] = \frac{1}{3}a^2h$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude. ◄

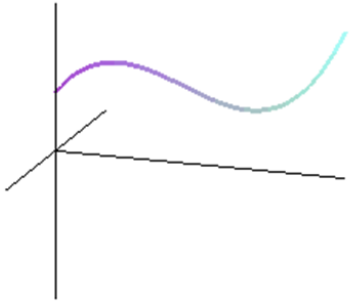
SOLIDS OF REVOLUTION

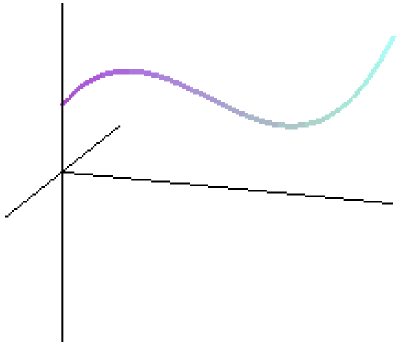
A *solid of revolution* is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the *axis of revolution*. Many familiar solids are of this type (Figure 6.2.8).

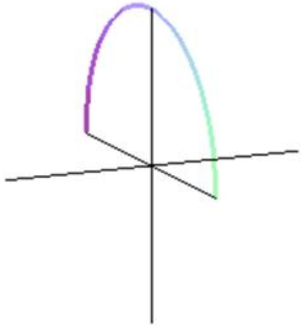


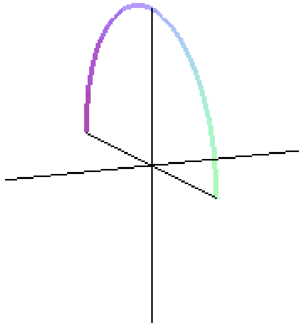
▶ **Figure 6.2.8**

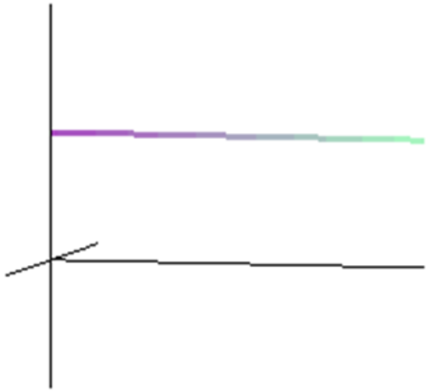
REVOLVE ABOUT X-AXIS

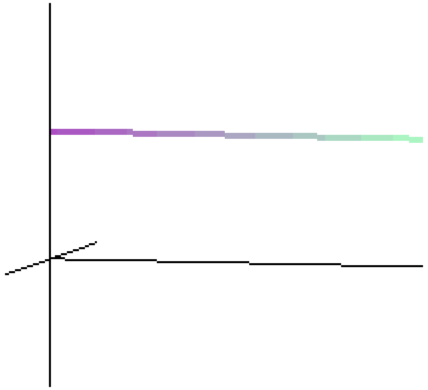


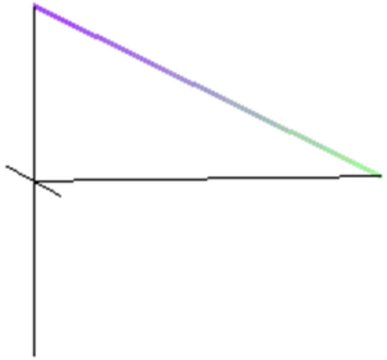


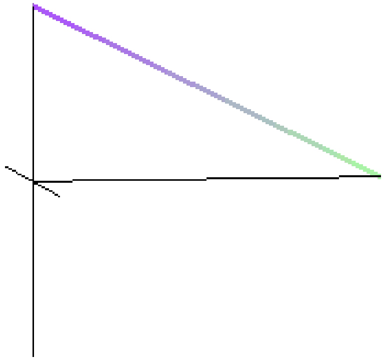




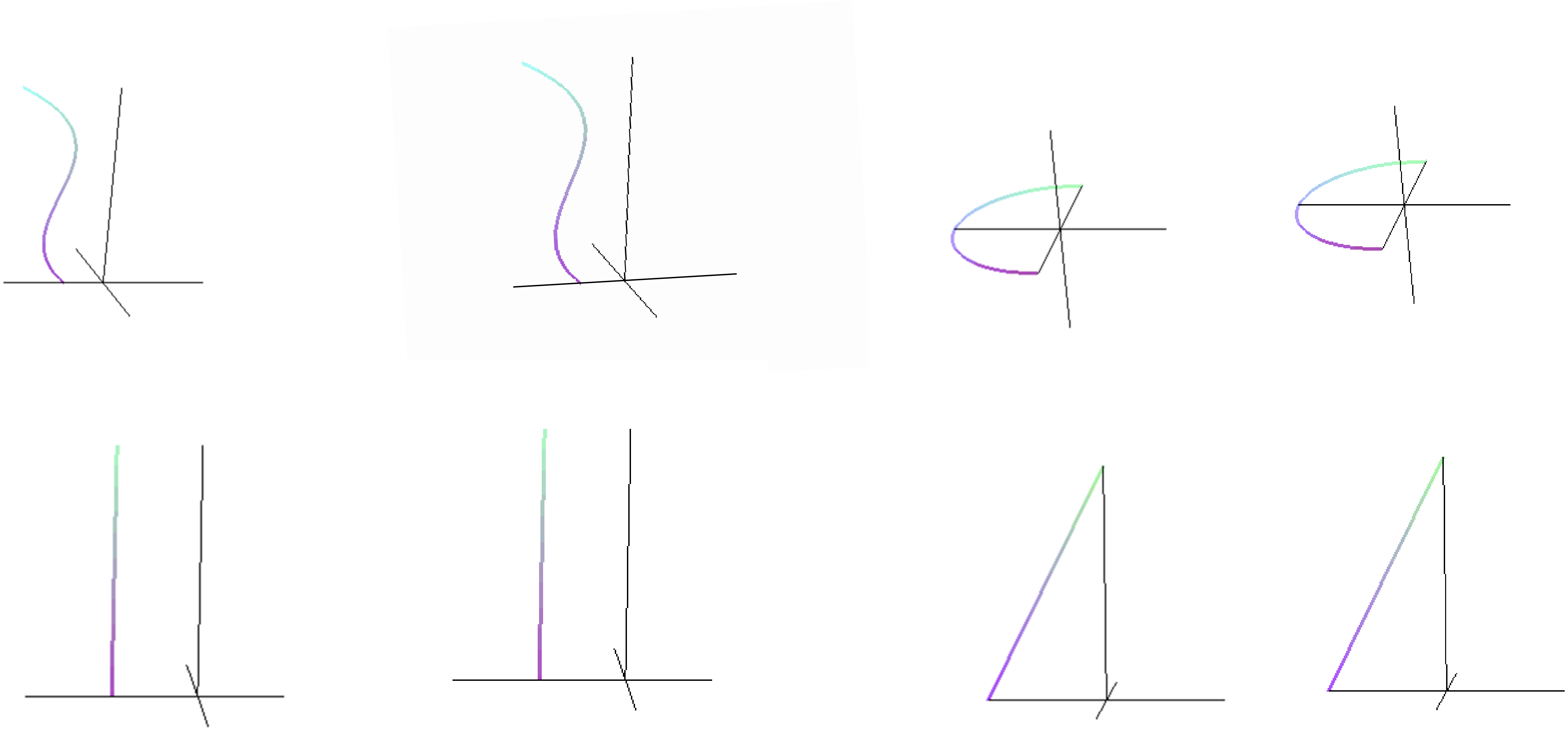




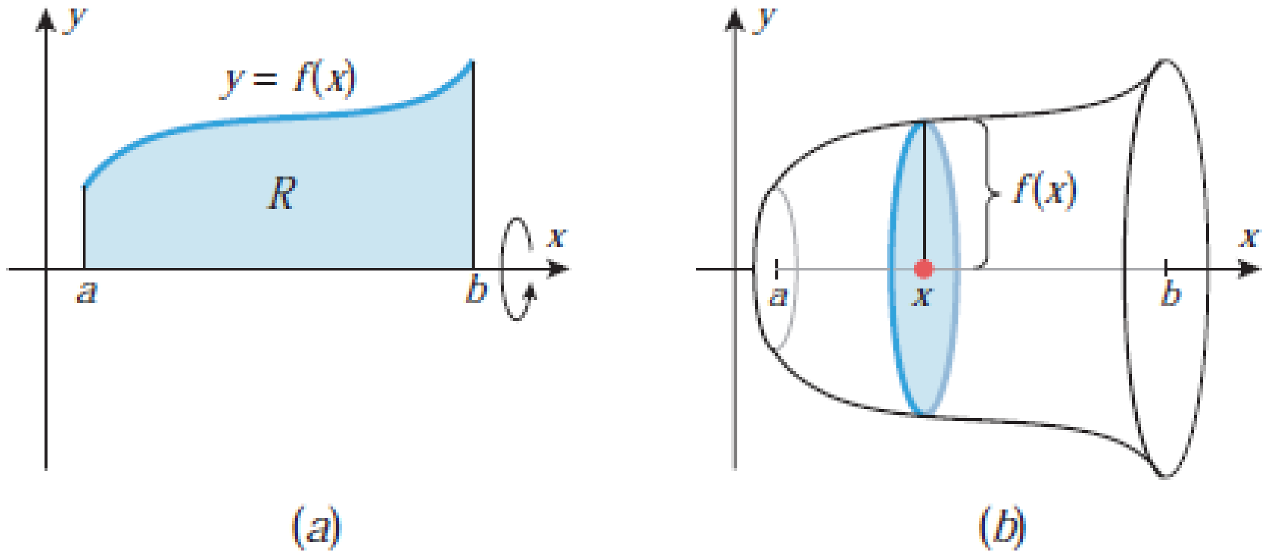




REVOLVE ABOUT Y-AXIS



6.2.4 PROBLEM Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

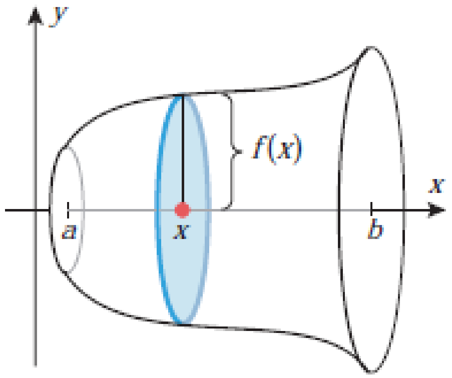


▶ **Figure 6.2.9**

INTEGRAL CALCULUS

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius $f(x)$ (Figure 6.2.9b). The area of this region is

$$A(x) = \pi[f(x)]^2$$



(b)

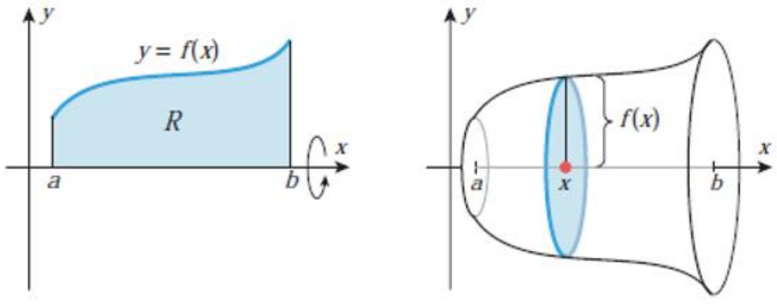
Thus, from (3) the volume of the solid is

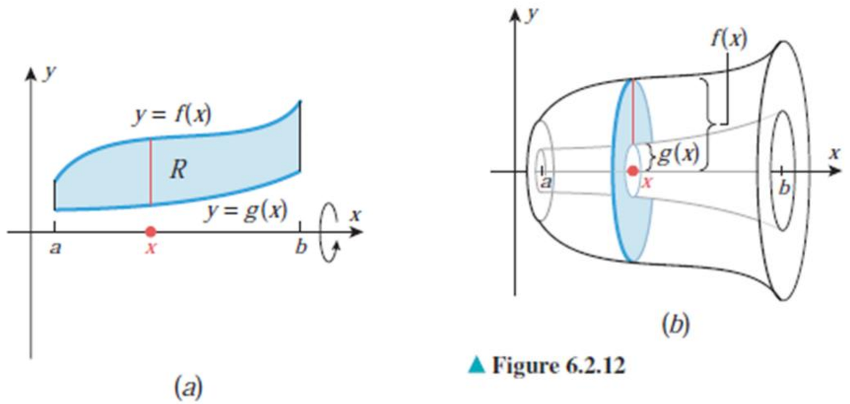
$$V = \int_a^b \pi[f(x)]^2\,dx \qquad (5)$$

Because the cross sections are disk shaped, the application of this formula is called the *method of disks*.

SUMMARY OF FORMULAS: VOLUME

VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE X -AXIS





▲ Figure 6.2.12

$$V = \int_a^b \pi[f(x)]^2\,dx$$

Volume formula by method of Disks

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

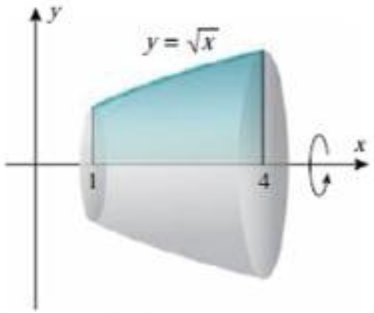
Volume formula by Method of washers

Finding volume by method of disk perpendicular to x-axis

Formula:

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

► **Example 1** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval [1, 4] is revolved about the x-axis (Figure 6.2.10).



▲ **Figure 6.2.10**

$f(x) = \sqrt{x}$

$a = 1 \quad b = 4$

$$V = \int_1^4 \pi (\sqrt{x})^2 \, dx$$

$$= \pi \int_1^4 x \, dx$$

$$= \pi \left. \frac{x^2}{2} \right|_1^4$$

$$= \left[\pi \frac{4^2}{2}\right] - \left(\pi \frac{1^2}{2}\right]$$

$$= 8\pi - \frac{\pi}{2} = \boxed{\frac{15\pi}{2}}$$

▶ **Example 2** Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

$f(x) = \frac{1}{2} + x^2 \qquad g(x) = x \qquad a = 0 \qquad b = 2$

$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx$$

$$= \pi \int_0^2 \left[\left(\tfrac{1}{2} + x^2\right)^2 - (x)^2\right] dx$$

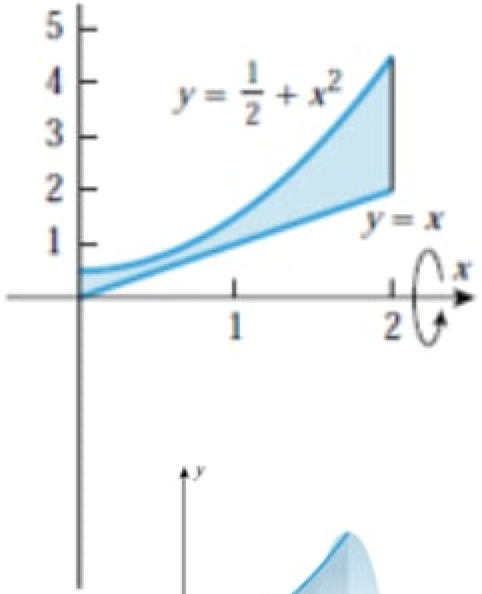
$$= \pi \int_0^2 \left(\tfrac{1}{4} + x^2 + x^4 - x^2\right) dx$$

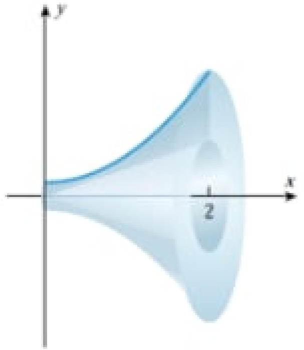
$$= \pi \int_0^2 \left(\tfrac{1}{4} + x^4\right) dx$$

$$= \pi \left(\tfrac{1}{4}x + \tfrac{x^5}{5}\right)\Big|_0^2$$

$$= \pi\left[\left(\tfrac{2}{4} + \tfrac{32}{5}\right) - 0\right]$$

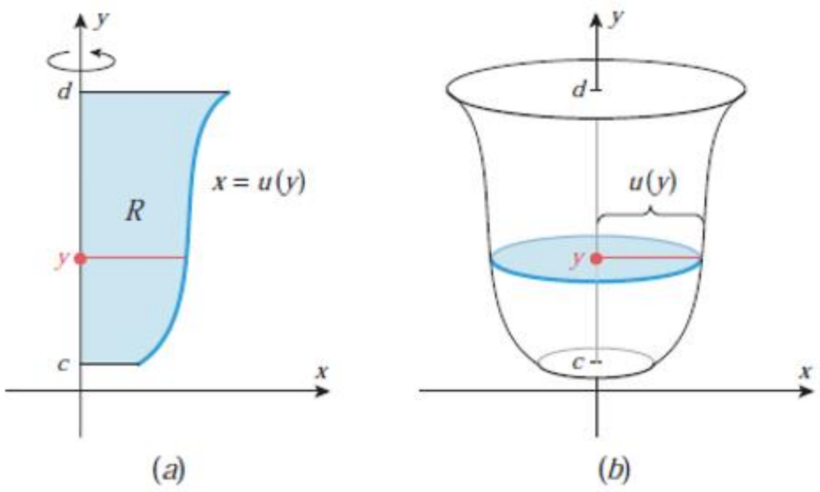
$$\boxed{V = \frac{69\pi}{10}}$$



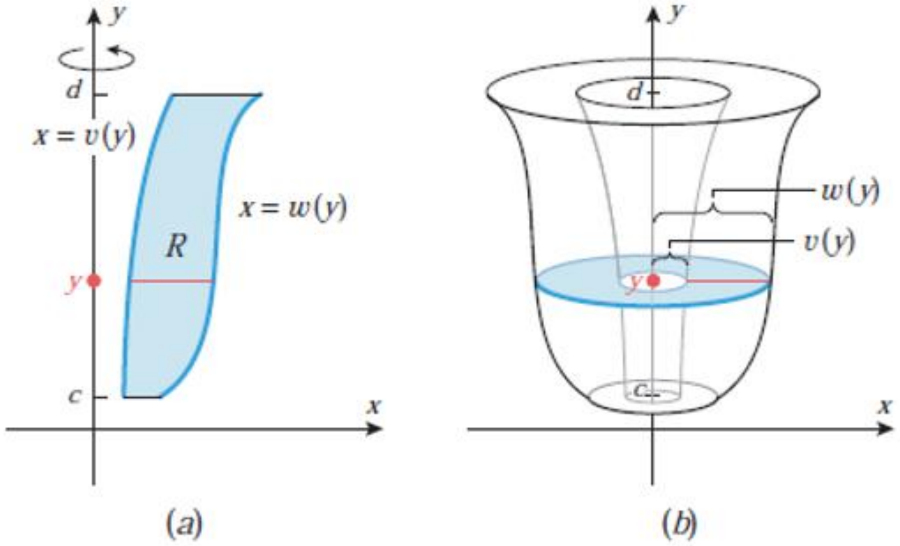


SUMMARY OF FORMULAS: VOLUME

VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE *y*-AXIS



Disks



Washers

$$V = \int_c^d \pi[u(y)]^2\,dy$$

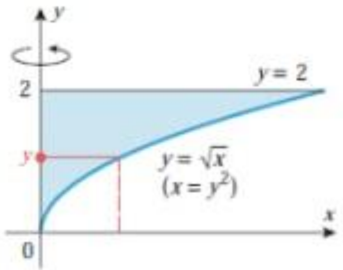
Disks

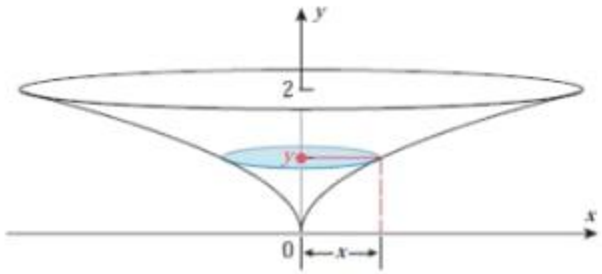
$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$$

Washers

Jun/09/2024

▶ **Example 5** Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.





$$V = \int_c^d \pi[u(y)]^2 \, dy$$

Disks

$y = \sqrt{x}$ $\quad y = 2 \quad y = \sqrt{x}$

$y^2 = x$ $\quad$ if $x = 0$

$x = y^2$ $\quad y = 0$

$u(y) = y^2$ $\quad c = 0 \quad d = 2$

$$V = \int_0^2 \pi [y^2]^2 dy$$

$$= \pi \int_0^2 y^4 dy$$

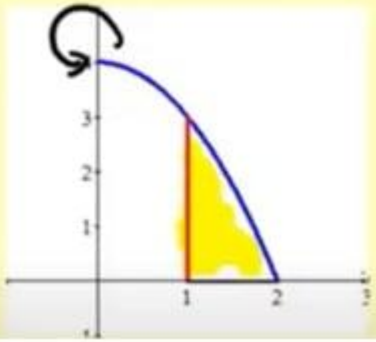
$$= \pi \left. \frac{y^5}{5} \right|_0^2$$

$$= \pi \frac{2^5}{5} - \pi \frac{0^5}{5}$$

$$V = \frac{32\pi}{5}$$

Determine the volume of the solid generated by the bounded region of given equation rotated about the y-axis.

$y = -x^2 + 4, x = 1, y = 0$



$y = -x^2 + 4 \quad x = 1$

$y - 4 = -x^2 \quad$ IF $x = 1$

$-y + 4 = x^2 \quad y = -1^2 + 4$

$=$

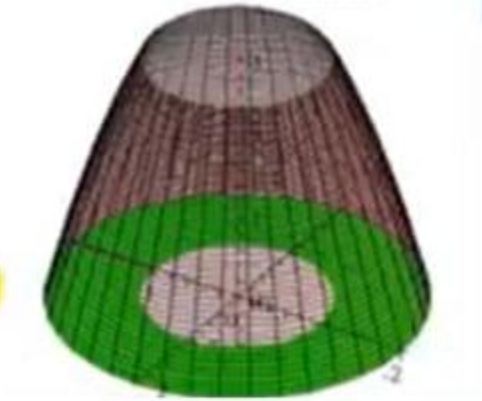
$4 - y = x^2$

$\sqrt{4-y} = x$

$x = \sqrt{4-y}$

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\, dy$$

Washers



$w(y) = \sqrt{4-y} \quad v(y) = 1$

$c = 0 \quad d = 3$

$V = \int_0^3 \pi [\sqrt{4-y}]^2 - [1]^2 dy$

$V = \pi \int_0^3 (4 - y - 1) dy$

$= \pi \int_0^3 (3 - y) dy$

$= \pi [3y - \frac{y^2}{2} |_0^3]$

$= \pi [(9 - \frac{9}{2}) - 0]$

$V = \frac{9\pi}{2}$

OTHER AXES OF REVOLUTION

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (3) and (4) and integrate an appropriate cross-sectional area to find the volume.

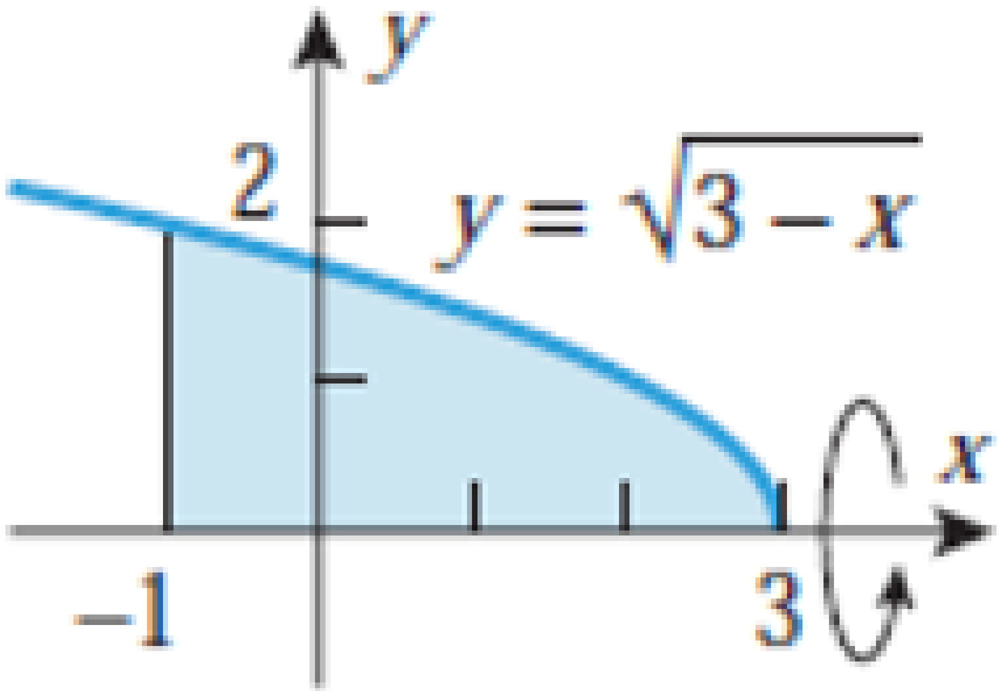
▶ **Example 6** Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval $[0, 2]$ is rotated about the line $y = -1$.

INTEGRAL CALCULUS

Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

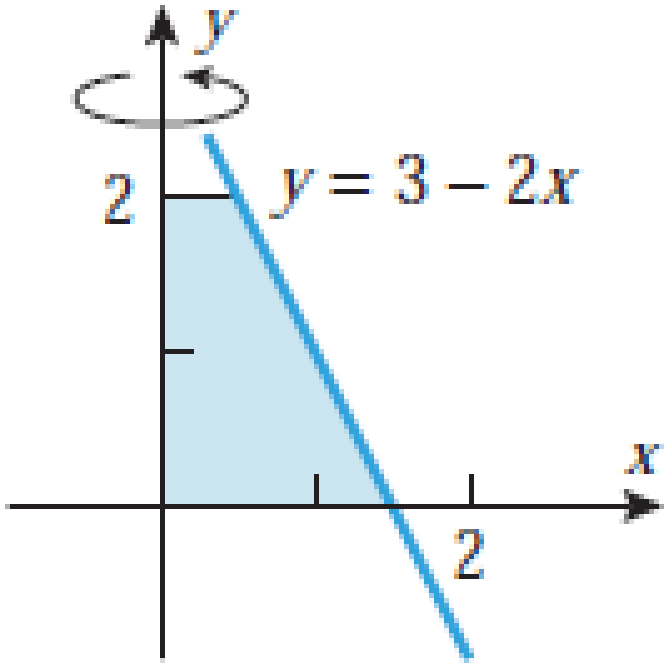
$$V = \int_a^b \pi[f(x)]^2\,dx$$

1.



1. $V = \pi \int_{-1}^{3} (3 - x)\,dx = 8\pi.$

3.



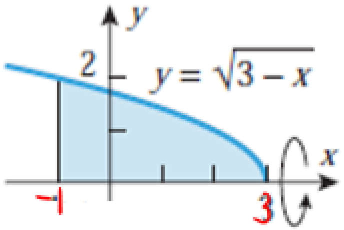
$$V = \int_c^d \pi[u(y)]^2\,dy$$

Disks

3. $V = \pi \int_0^2 \frac{1}{4}(3-y)^2\,dy = 13\pi/6.$

INTEGRAL CALCULUS

1.



$V = \int_a^b \pi[f(x)]^2\,dx$: $f(x) = \sqrt{3-x}$

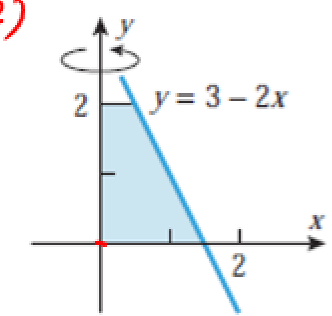
$$V = \int_{-1}^{3} \pi\left(\sqrt{3-x}\right)^2 dx$$

$$= \pi\int_{-1}^{3} (3-x)\,dx$$

$$= \pi\left(\left[3x - \frac{x^2}{2}\right]_{-1}^{3}\right) = \pi\left[\left(9-\frac{9}{2}\right) - \left(-3-\frac{1}{2}\right)\right]$$

$$V = 8\pi$$

2)



$x = \frac{3-y}{2}$

$x = \frac{1}{2}(3-y)$

$V = \int_c^d \pi[u(y)]^2\,dy$

$$V = \int_0^2 \pi\left(\frac{1}{2}(3-y)\right)^2 dy$$

$$= \pi\int_0^2 \frac{1}{4}(3-y)^2\,dy$$

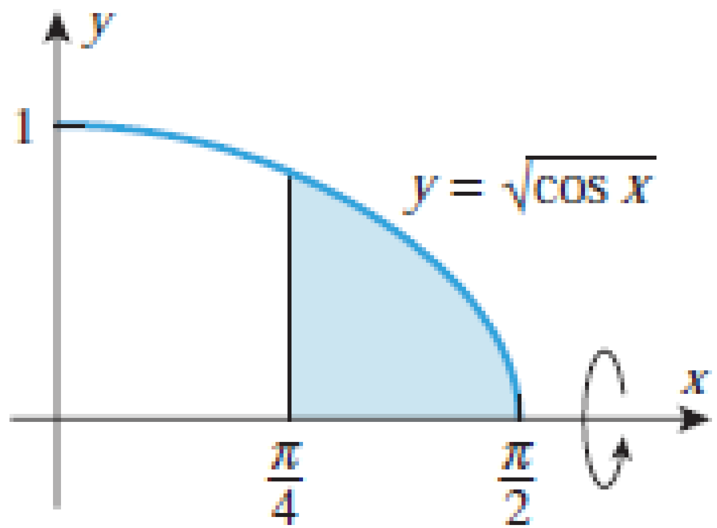
$$= \frac{\pi}{4}\int_0^2 (9-6y+y^2)\,dy$$

$$= \frac{\pi}{4}\left(9y - 3y^2 + \frac{y^3}{3}\Big]_0^2\right)$$

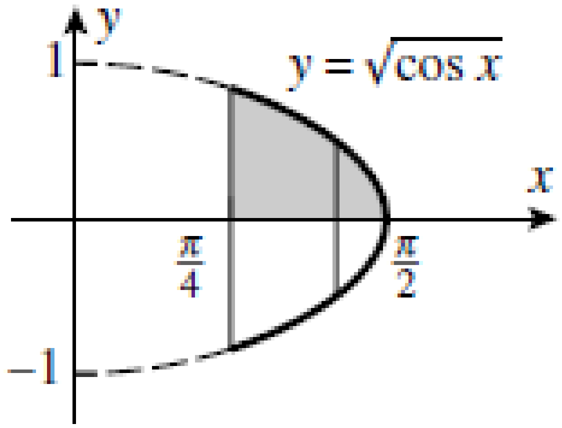
$$= \frac{\pi}{4}\left(\left(18-12+\frac{8}{3}\right)-0\right)$$

$$= \frac{\pi}{4}\left(\frac{26}{3}\right) = \frac{13\pi}{6}$$

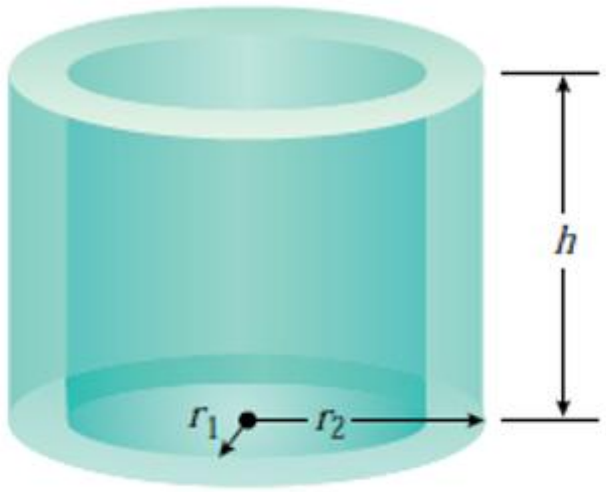
5.



5. $V = \pi \int_{\pi/4}^{\pi/2} \cos x \, dx = (1 - \sqrt{2}/2)\pi.$



VOLUMES BY CYLINDRICAL SHELLS



▲ **Figure 6.3.2**

A *cylindrical shell* is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h can be written as

$$\begin{aligned} V &= [\text{area of cross section}] \cdot [\text{height}] \\ &= (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \cdot \left[\tfrac{1}{2}(r_1 + r_2)\right] \cdot h \cdot (r_2 - r_1) \end{aligned}$$

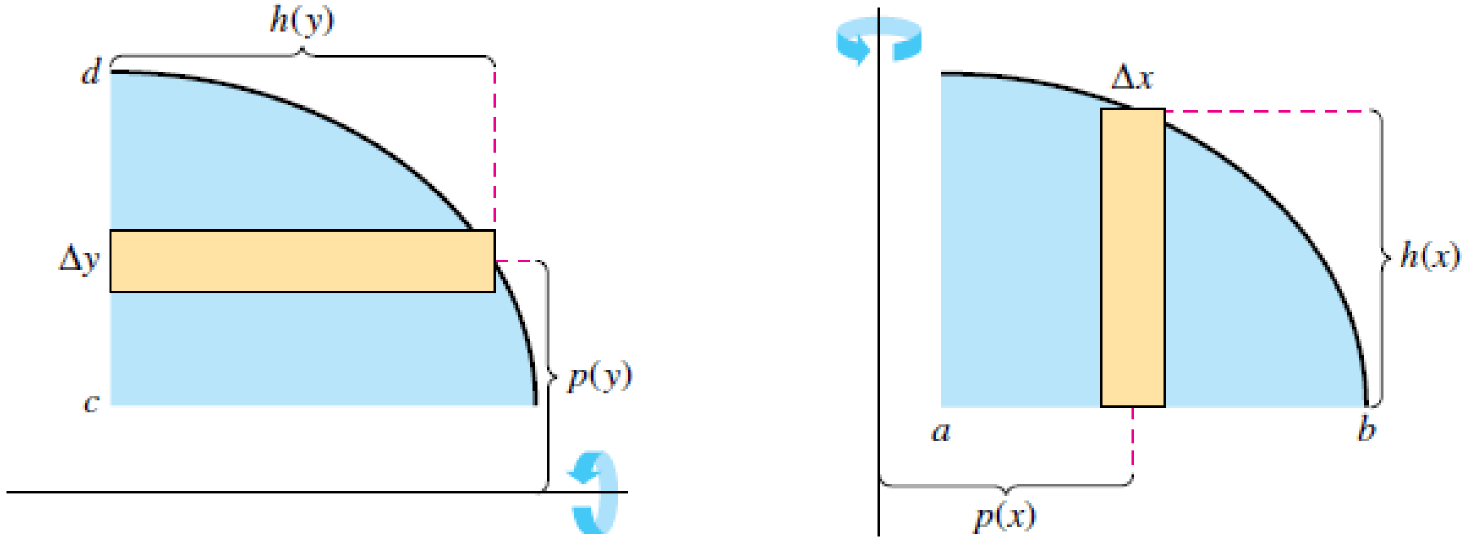
But $\frac{1}{2}(r_1 + r_2)$ is the average radius of the shell and $r_2 - r_1$ is its thickness, so

$$V = 2\pi \cdot [\text{average radius}] \cdot [\text{height}] \cdot [\text{thickness}] \qquad (1)$$

THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the formulas below. (See Figure 7.29.)

Horizontal Axis of Revolution	**Vertical Axis of Revolution**
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\, dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\, dx$



Horizontal axis of revolution

Vertical axis of revolution

Figure 7.29

EXAMPLE 1 Using the Shell Method to Find Volume

Find the volume of the solid formed by revolving the region bounded by

$$y = x - x^3$$

and the x-axis $(0 \le x \le 1)$ about the y-axis.

Solution Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 7.30. The width Δx indicates that x is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(x) = x$, and the height of the rectangle is

$$h(x) = x - x^3.$$

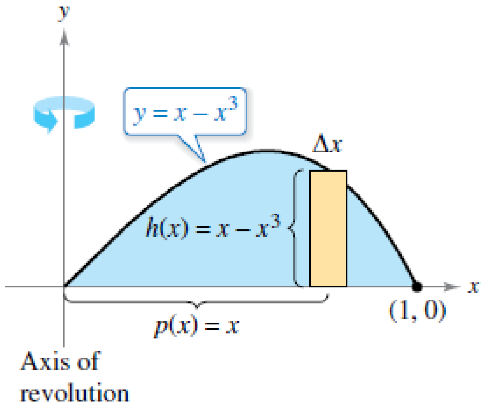


Figure 7.30

Because x ranges from 0 to 1, apply the shell method to find the volume of the solid.

$$V = 2\pi \int_a^b p(x)h(x)\,dx$$

$$= 2\pi \int_0^1 x(x - x^3)\,dx$$

$$= 2\pi \int_0^1 (-x^4 + x^2)\,dx \qquad \text{Simplify.}$$

$$= 2\pi \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_0^1 \qquad \text{Integrate.}$$

$$= 2\pi \left(-\frac{1}{5} + \frac{1}{3} \right)$$

$$= \frac{4\pi}{15}$$

EXAMPLE 2 Using the Shell Method to Find Volume

Find the volume of the solid formed by revolving the region bounded by the graph of

$$x = e^{-y^2}$$

and the y-axis $(0 \le y \le 1)$ about the x-axis.

Solution Because the axis of revolution is horizontal, use a horizontal representative rectangle, as shown in Figure 7.31. The width Δy indicates that y is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(y) = y$, and the height of the rectangle is $h(y) = e^{-y^2}$. Because y ranges from 0 to 1, the volume of the solid is

$$V = 2\pi \int_c^d p(y)h(y)\,dy \qquad \text{Apply shell method.}$$

$$= 2\pi \int_0^1 ye^{-y^2}\,dy$$

$$= -\pi\Big[e^{-y^2}\Big]_0^1 \qquad \text{Integrate.}$$

x

$$= \pi\left(1 - \frac{1}{e}\right)$$

$$\approx 1.986.$$

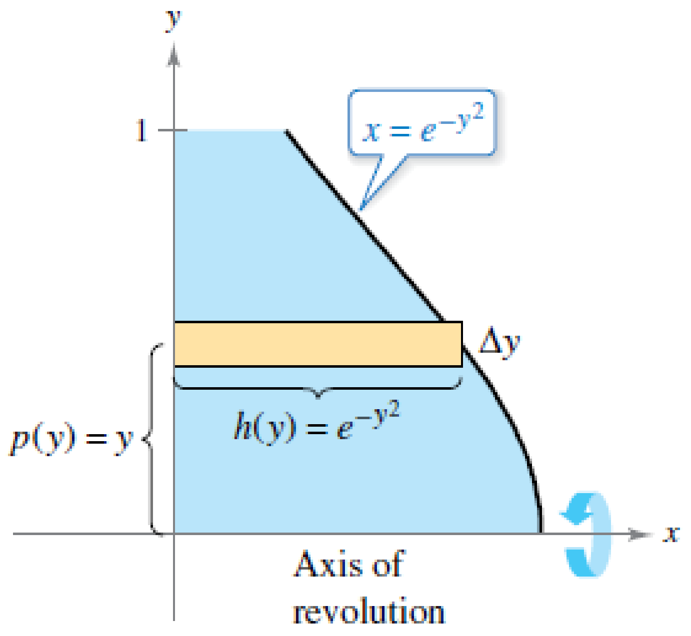


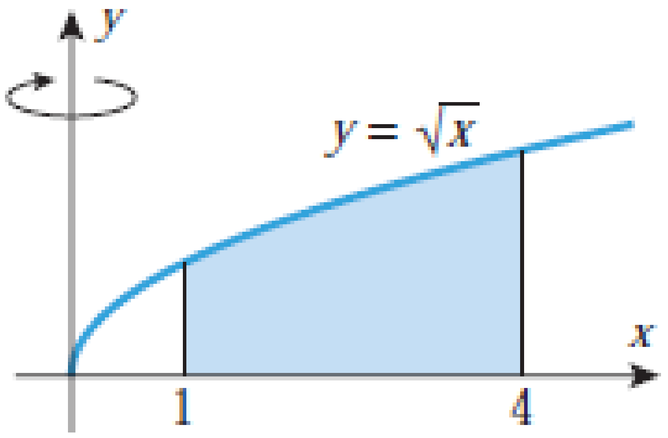
Figure 7.31

In summary, we have the following result.

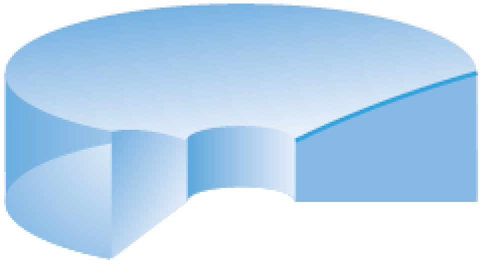
6.3.2 VOLUME BY CYLINDRICAL SHELLS ABOUT THE y-AXIS Let f be continuous and nonnegative on $[a, b]$ $(0 \le a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the y-axis is given by

$$V = \int_a^b 2\pi x f(x)\, dx \qquad (2)$$

▶ **Example 1** Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis.



(a)



Cutaway view of the solid

(b)

▲ **Figure 6.3.6**

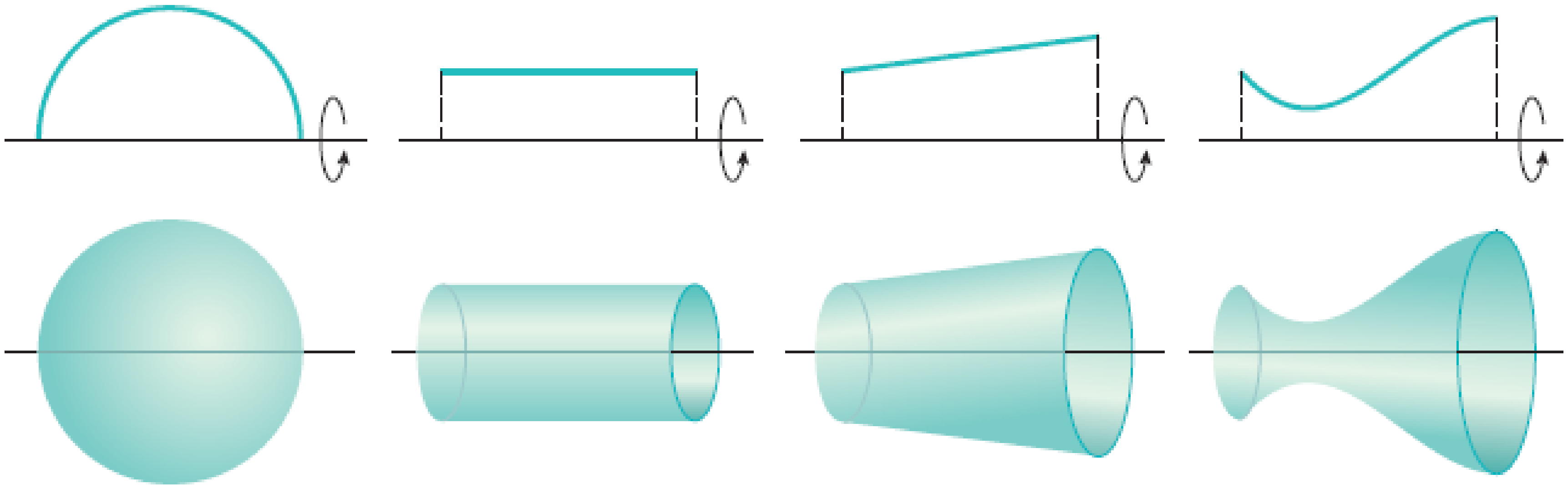
Solution. First sketch the region (Figure 6.3.6a); then imagine revolving it about the y-axis (Figure 6.3.6b). Since $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$, Formula (2) yields

$$V = \int_1^4 2\pi x\sqrt{x}\,dx = 2\pi \int_1^4 x^{3/2}\,dx = \left[2\pi \cdot \frac{2}{5}x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32 - 1] = \frac{124\pi}{5}$$ ◀

SURFACE AREA

A *surface of revolution* is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).

Some Surfaces of Revolution



INTEGRAL CALCULUS

6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

This result provides both a definition and a formula for computing surface areas. Where convenient, this formula can also be expressed as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (4)$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis can be expressed as

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (5)$$

Example 1 Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

Solution. First sketch the curve; then imagine revolving it about the x-axis (Figure 6.5.6). Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

$$S = \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi\int_0^1 x^3(1+9x^4)^{1/2}\,dx$$

$$= \frac{2\pi}{36}\int_1^{10} u^{1/2}\,du \qquad \begin{array}{l} u = 1+9x^4 \\ du = 36x^3\,dx \end{array}$$

$$= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2}-1) \approx 3.56$$

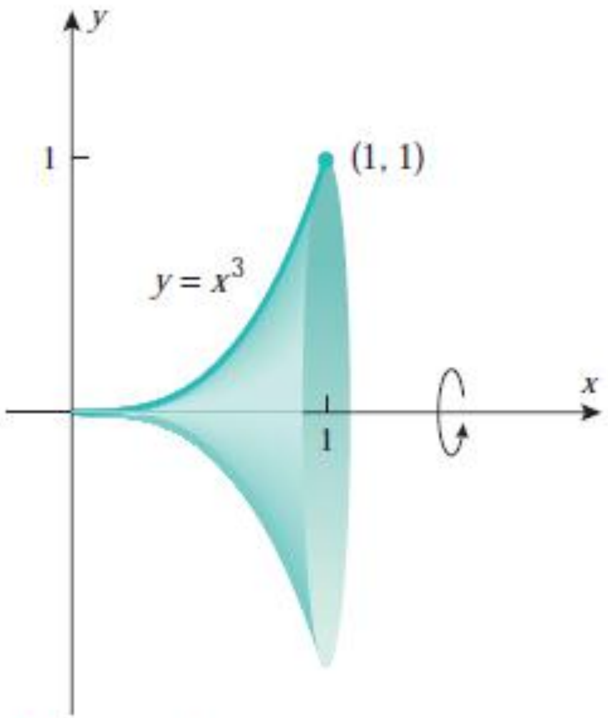


Figure 6.5.6

Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.

Solution. First sketch the curve; then imagine revolving it about the y-axis (Figure 6.5.7). Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and $x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

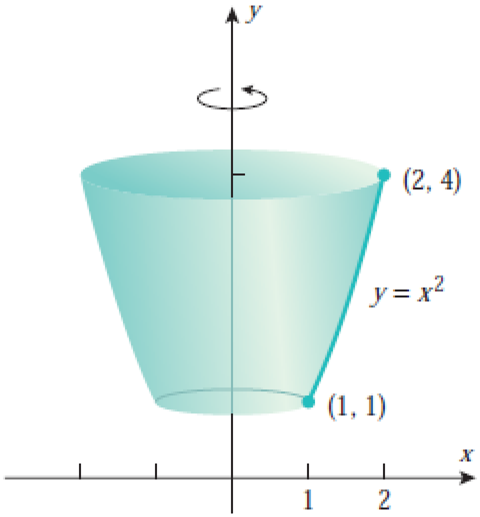
$$S = \int_1^4 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^4 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \pi \int_1^4 \sqrt{4y+1}\, dy$$

$$= \frac{\pi}{4}\int_5^{17} u^{1/2}\, du \qquad \begin{array}{l} u = 4y+1 \\ du = 4\,dy \end{array}$$

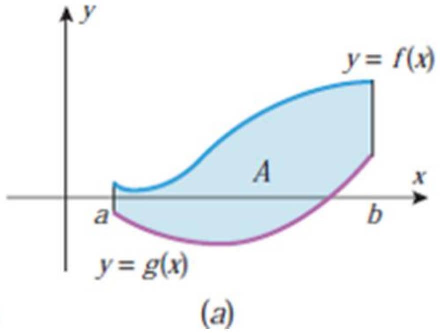
$$= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85$$



▲ Figure 6.5.7

SUMMARY OF FORMULAS:
AREAS

$$A = \int_a^b [f(x) - g(x)]\,dx$$



▶ Figure 6.1.3

$$A = \int_c^d [w(y) - v(y)]\,dy$$

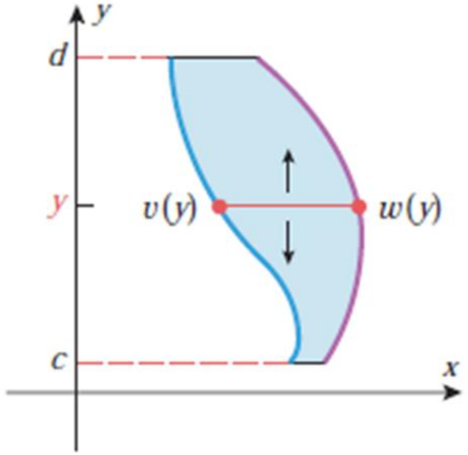
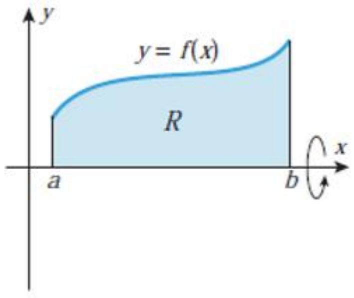
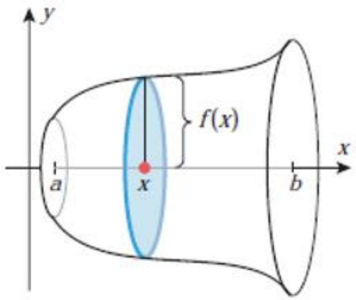


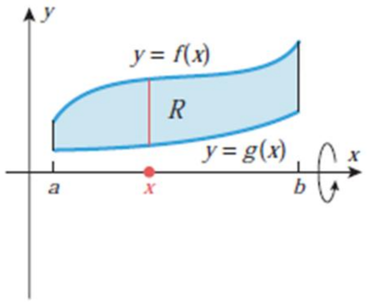
Figure 6.1.12

SUMMARY OF FORMULAS: VOLUME

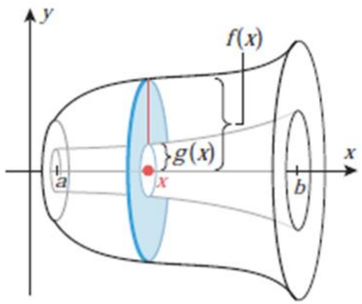
VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE X -AXIS







(a)



(b)

▲ Figure 6.2.12

$$V = \int_a^b \pi[f(x)]^2\,dx$$

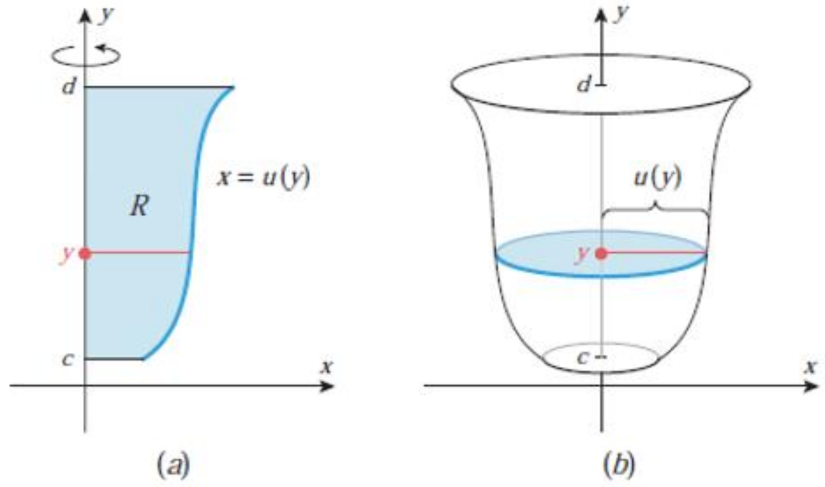
Volume formula by method of Disks

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

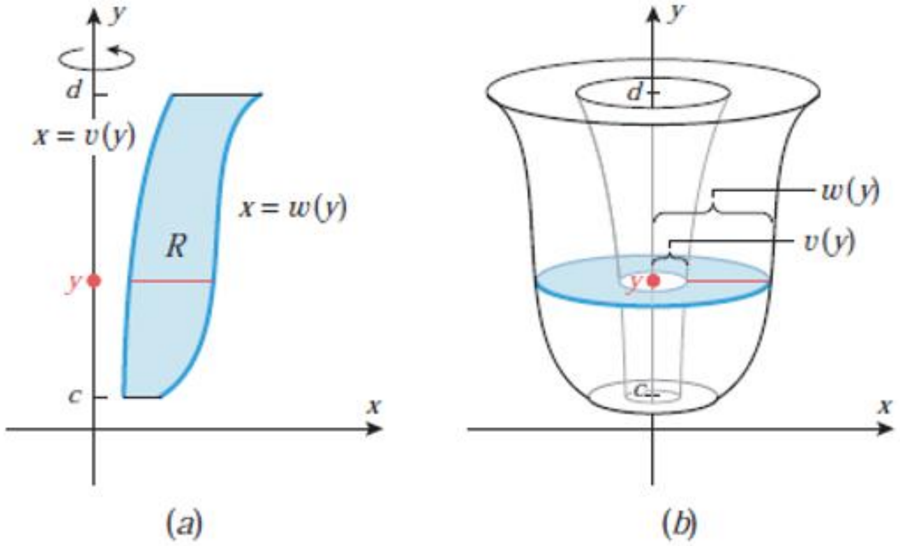
Volume formula by Method of washers

SUMMARY OF FORMULAS: VOLUME

VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE *y*-AXIS



Disks



Washers

$$V = \int_c^d \pi[u(y)]^2\,dy$$

Disks

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$$

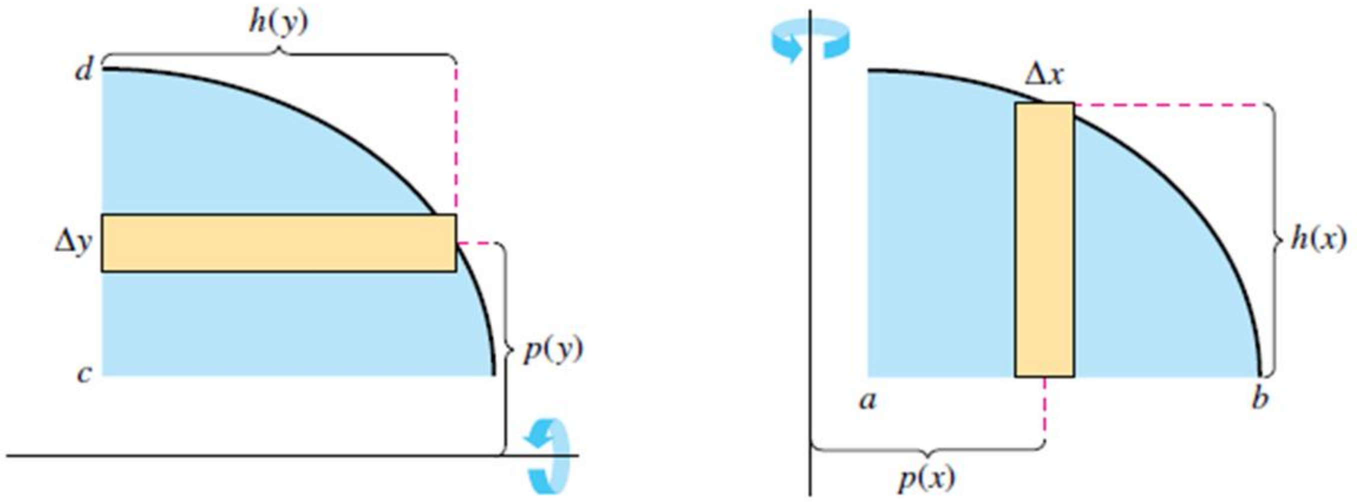
Washers

SUMMARY OF FORMULAS: VOLUME

THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the formulas below. (See Figure 7.29.)

Horizontal Axis of Revolution	**Vertical Axis of Revolution**
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$



Horizontal axis of revolution   Vertical axis of revolution

Figure 7.29