TRIGONOMETRIC TRANSFORMATION
WALLI'S FORMULA

Walli' sFormula

$$\int_0^{\frac{\pi}{2}} sin^m \theta \, cos^n \theta \; d\theta = \frac{(m-1)(m-3)......(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4).........} \bullet \alpha$$

$$where \quad \begin{cases} \alpha = \dfrac{\pi}{2} \quad when\, m\, and\, n\, are\, even \\ otherwise \quad \alpha = 1 \end{cases}$$

INTEGRAL CALCULUS

EXAMPLE Evaluate the following integrals.

$$1.\quad \int_0^{\frac{\pi}{2}} 3\cos^8 x \sin^2 x \, dx = 3\frac{(1)(7)(5)(3)(1)}{(10)(8)(6)(4)(2)} \bullet \frac{\pi}{2} = \frac{21\pi}{512} = 0.12885$$

$$2.\quad \int_0^{\frac{\pi}{2}} 8\cos^7 x \sin^3 x \, dx = 8\frac{(2)(6)(4)(2)}{(10)(8)(6)(4)(2)} \bullet 1 = \frac{1}{5} = 0.2$$

$$3.\quad \int_0^{\frac{\pi}{6}} \cos^8 3x = \frac{1}{3} \bullet \frac{(7)(5)(3)(1)}{(8)(6)(4)(2)} \bullet \frac{\pi}{2} = \frac{35\pi}{768} = 0.143$$

$$let u = 3x; \qquad du = 3dx$$

$$when \ x = 0; \quad u = 0$$

$$when \ x = \frac{\pi}{6}; \ u = \frac{3\pi}{6} = \frac{\pi}{2}$$

INTEGRAL CALCULUS

EXAMPLE Evaluate the following integrals.

$$4.\ \int_0^{\frac{\pi}{4}} \sin^7 2x = \frac{1}{2} \bullet \frac{(6)(4)(2)}{(7)(5)(3)(1)} \bullet 1 = 0.229$$

$$let u = 2x; \qquad du = 2dx$$

$$when\ \ x = 0; \quad u = 0$$

$$when\ \ x = \frac{\pi}{4};\ u = \frac{2\pi}{4} = \frac{\pi}{2}$$

INTEGRAL CALCULUS

EXAMPLE Evaluate the following integrals.

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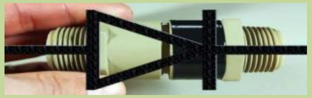
CALCULUS 2
LESSON 10

INTEGRATION TECHNIQUES

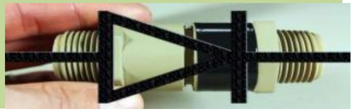
Integration by Trigonometric Substitution

Integration by Partial Fraction expansion

PRESENTED BY ENGR. JOHN R. REJANO,ECE



OBJECTIVES



- to evaluate integrals using integration by substitution
- to evaluate integrals using integration by give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions

TRIGONOMETRIC SUBSTITUTIONS

THE METHOD OF TRIGONOMETRIC SUBSTITUTION

To start, we will be concerned with integrals that contain expressions of the form

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}$$

in which a is a positive constant. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical. For example, to eliminate the radical in the expression $\sqrt{a^2 - x^2}$, we can make the substitution

$$x = a\sin\theta, \quad -\pi/2 \le \theta \le \pi/2 \tag{1}$$

which yields

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)}$$

$$= a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta$$

$\cos\theta \ge 0$ since $-\pi/2 \le \theta \le \pi/2$

The restriction on θ in (1) serves two purposes—it enables us to replace $|\cos\theta|$ by $\cos\theta$ to simplify the calculations, and it also ensures that the substitutions can be rewritten as $\theta = \sin^{-1}(x/a)$, if needed.

▶ Example 1 Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. To eliminate the radical we make the substitution

$$x = 2\sin\theta, \quad dx = 2\cos\theta\, d\theta$$

This yields

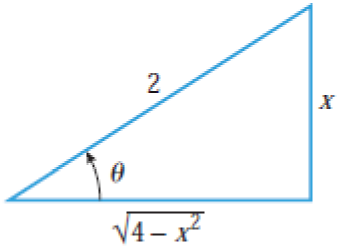
$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C$$

At this point we have completed the integration; however, because the original integral was expressed in terms of x, it is desirable to express $\cot\theta$ in terms of x as well. This can be done using trigonometric identities, but the expression can also be obtained by writing the substitution $x = 2\sin\theta$ as $\sin\theta = x/2$ and representing it geometrically as in Figure 7.4.1. From that figure we obtain

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$



$x = 2\sin\theta$

▲ Figure 7.4.1

x

Substituting this in (2) yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$ ◀

EXAMPLE 1 **Trigonometric Substitution: $u = a \sin \theta$**

Find $\displaystyle\int \frac{dx}{x^2\sqrt{9 - x^2}}$.

Solution First, note that the basic integration rules do not apply. To use trigonometric substitution, you should observe that

$$\sqrt{9 - x^2}$$

is of the form $\sqrt{a^2 - u^2}$. So, you can use the substitution

$$x = a \sin \theta = 3 \sin \theta.$$

Using differentiation and the triangle shown in Figure 8.6, you obtain

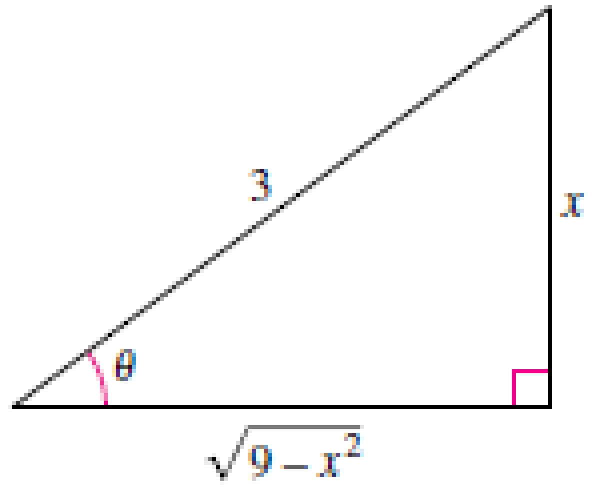
$$dx = 3 \cos \theta \, d\theta, \quad \sqrt{9 - x^2} = 3 \cos \theta, \quad \text{and} \quad x^2 = 9 \sin^2 \theta.$$

So, trigonometric substitution yields

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9 - x^2}} &= \int \frac{3 \cos \theta \, d\theta}{(9 \sin^2 \theta)(3 \cos \theta)} && \text{Substitute.}\\
&= \frac{1}{9}\int \frac{d\theta}{\sin^2 \theta} && \text{Simplify.}\\
&= \frac{1}{9}\int \csc^2 \theta \, d\theta && \text{Trigonometric identity}\\
&= -\frac{1}{9}\cot \theta + C && \text{Apply Cosecant Rule.}\\
&= -\frac{1}{9}\left(\frac{\sqrt{9 - x^2}}{x}\right) + C && \text{Substitute for } \cot \theta.\\
&= -\frac{\sqrt{9 - x^2}}{9x} + C.
\end{aligned}$$

Note that the triangle in Figure 8.6 can be used to convert the θ's back to x's, as shown.

$$\begin{aligned}
\cot \theta &= \frac{\text{adj.}}{\text{opp.}}\\
&= \frac{\sqrt{9 - x^2}}{x}
\end{aligned}$$



$\sin \theta = \dfrac{x}{3}$, $\cot \theta = \dfrac{\sqrt{9 - x^2}}{x}$

Figure 8.6

EXAMPLE 2 Trigonometric Substitution: $u = a \tan \theta$

Find $\displaystyle\int \frac{dx}{\sqrt{4x^2 + 1}}$.

Solution Let $u = 2x$, $a = 1$, and $2x = \tan \theta$, as shown in Figure 8.7. Then

$$dx = \frac{1}{2}\sec^2 \theta \, d\theta \quad \text{and} \quad \sqrt{4x^2 + 1} = \sec \theta.$$

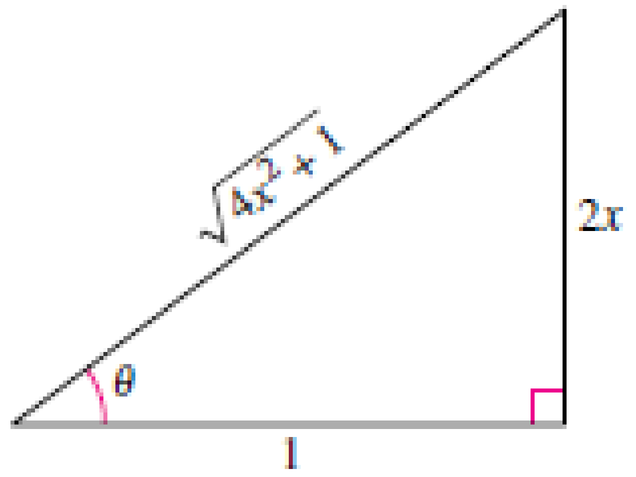
Trigonometric substitution produces

$$\int \frac{dx}{\sqrt{4x^2 + 1}} = \frac{1}{2}\int \frac{\sec^2 \theta \, d\theta}{\sec \theta} \qquad \text{Substitute.}$$

$$= \frac{1}{2}\int \sec \theta \, d\theta \qquad \text{Simplify.}$$

$$= \frac{1}{2}\ln|\sec \theta + \tan \theta| + C \qquad \text{Apply Secant Rule.}$$

$$= \frac{1}{2}\ln\left|\sqrt{4x^2 + 1} + 2x\right| + C. \qquad \text{Back-substitute.}$$



$\tan \theta = 2x$, $\sec \theta = \sqrt{4x^2 + 1}$

Figure 8.7

Ex.1 $\int \frac{dx}{x^2\sqrt{4-x^2}}$

$a^2 - x^2$

$a = 2 \quad u = x$

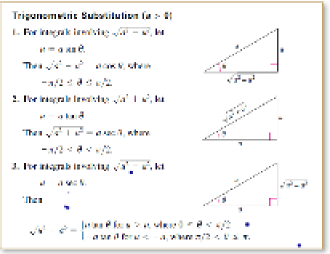


Table 7.4.1

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

by p.t.

$\cos^2\theta = 1 - \sin^2\theta$

$= \int \frac{dx}{(x)^2\sqrt{2^2 - (x)^2}}$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4 - (2\sin\theta)^2}}$

$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4 - 4\sin^2\theta}}$

$\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4(1-\sin^2\theta)}}$

P.T

$\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}}$

$\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\; 2\cos\theta}$

$\frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$

$\frac{1}{4}\int \csc^2\theta\, d\theta$

$\frac{1}{4}(-\cot\theta) + c$

$-\frac{1}{4}\cot\theta + c$

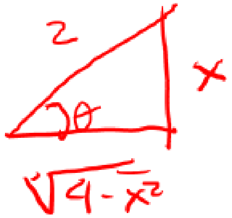
$x = 2\sin\theta$

$\frac{1}{\sin\theta} = \csc$

$\frac{1}{\sin^2\theta} = \csc^2\theta$

$\frac{x}{2} = \sin\theta$

$\sin\theta = \frac{x}{2}$



$c^2 = a^2 + b^2$

$2^2 = a^2 + x^2$

$2^2 - x^2 = a^2$

$4 - x^2 = a^2$

$\sqrt{4-x^2} = a$

$\tan = \frac{opp}{adj}$ $\quad \cot = \frac{adj}{opp}$ $\quad \cot\theta = \frac{\sqrt{4-x^2}}{x}$

$= \boxed{-\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C}$ or $\boxed{\frac{-\sqrt{4-x^2}}{4x} + C}$

Ex 2.

Find $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

$$\int \frac{dx}{(x)^2\sqrt{3^2-(x)^2}}$$

$a=3 \quad u=x$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\int \frac{3\cos\theta\, d\theta}{(3\sin\theta)^2\sqrt{9-(3\sin\theta)^2}}$$

$$\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}$$

$$\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9(1-\sin^2\theta)}}$$

$$\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9\cos^2\theta}}$$

$$\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta \; 3\cos\theta}$$

$$\frac{1}{9}\int \frac{d\theta}{\sin^2\theta}$$

$$\frac{1}{9}\int \csc^2\theta\, d\theta$$

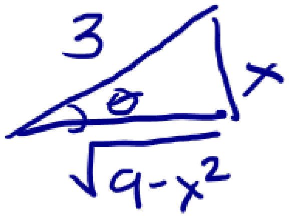
$$= \frac{1}{9}\,-\cot\theta + c$$

$$= -\frac{1}{9}\cot\theta + c$$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

$\sin\theta = \frac{x}{3}$



$\cot\theta = \frac{adj}{opp}$

$\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$$= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C$$

or

$$\frac{-\sqrt{9-x^2}}{9x} + C$$

3) $\int \frac{dx}{\sqrt{4+x^2}}$

$$\int \frac{dx}{\sqrt{2^2+(x)^2}}$$

$a=2 \quad u=x$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+(2\tan\theta)^2}}$$

$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4(1+\tan^2\theta)}}$$ ← pt $\sec^2\theta = \tan^2\theta + 1 = 1+\tan^2\theta$

$$= \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4\sec^2\theta}}$$

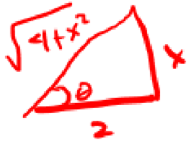
$$= \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta \qquad \tan\theta = \frac{x}{2}$



$\tan\theta = \frac{x}{2}$ $\qquad \sec\theta = \frac{1}{\cos\theta}$ $\qquad \cos = \frac{adj}{hyp}$

$\sec = \frac{hyp}{adj}$

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2} + x}{2}\right| + c$$

1) $\int \frac{dx}{\sqrt{4x^2+1}}$

$x = a\tan\theta$

$\int \frac{dx}{\sqrt{(2x)^2+(1)^2}}$

$u = 2x \quad a = 1$

$2x = \tan\theta$

$2\,dx = \sec^2\theta\, d\theta$

$dx = \frac{\sec^2\theta\, d\theta}{2}$

$\int \frac{\sec^2\theta\, d\theta \,/\, 2}{\sqrt{(\tan\theta)^2+1}}$

$\frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sqrt{\tan^2\theta+1}} \rightarrow PT$

$\frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}}$

$\frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec\theta}$

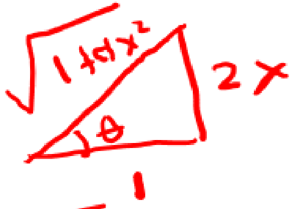
$\frac{1}{2}\int \sec\theta\, d\theta$

$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

$c^2 = a^2 + b^2$

$2x = \tan\theta$

$\tan\theta = \frac{2x}{1}$



$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{1+4x^2}}{1}$

$= \frac{1}{2}\ln\left|\sqrt{1+4x^2} + 2x\right| + C$

Evaluate $\int \frac{\sqrt{x^2 - 25}}{x}\,dx$, assuming that $x \geq 5$.

$a = 5 \qquad x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{(5\sec\theta)^2 - 25}}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{25(\sec^2\theta - 1)}}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{25\tan^2\theta}}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$$

$$= \int 5\tan\theta\tan\theta\,d\theta$$

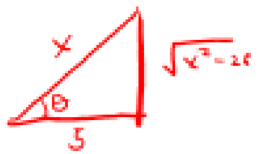
$$= 5\int \tan^2\theta\,d\theta$$

$$= 5\int (\sec^2\theta - 1)\,d\theta$$

$$= 5\tan\theta - 5\theta + C$$

$$= 5\,\frac{\sqrt{x^2-25}}{5} - 5\sec^{-1}\frac{x}{5} + C$$

$$= \sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + C$$

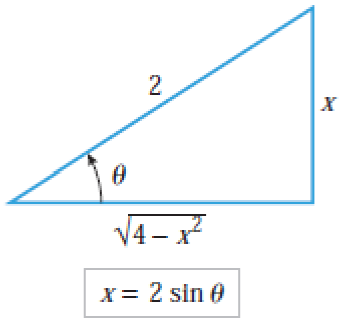


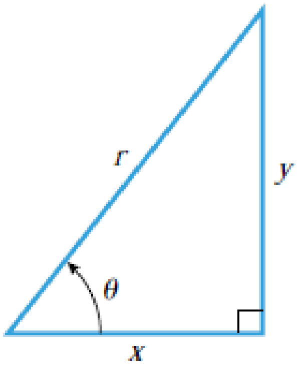
$x = 5\sec\theta$

$\sec\theta = \frac{x}{5} \qquad s = \frac{H}{A}$

$\theta = \sec^{-1}\frac{x}{5}$

$\tan\theta = \frac{opp}{adj} = \frac{\sqrt{x^2-25}}{5}$





▲ Figure B.6

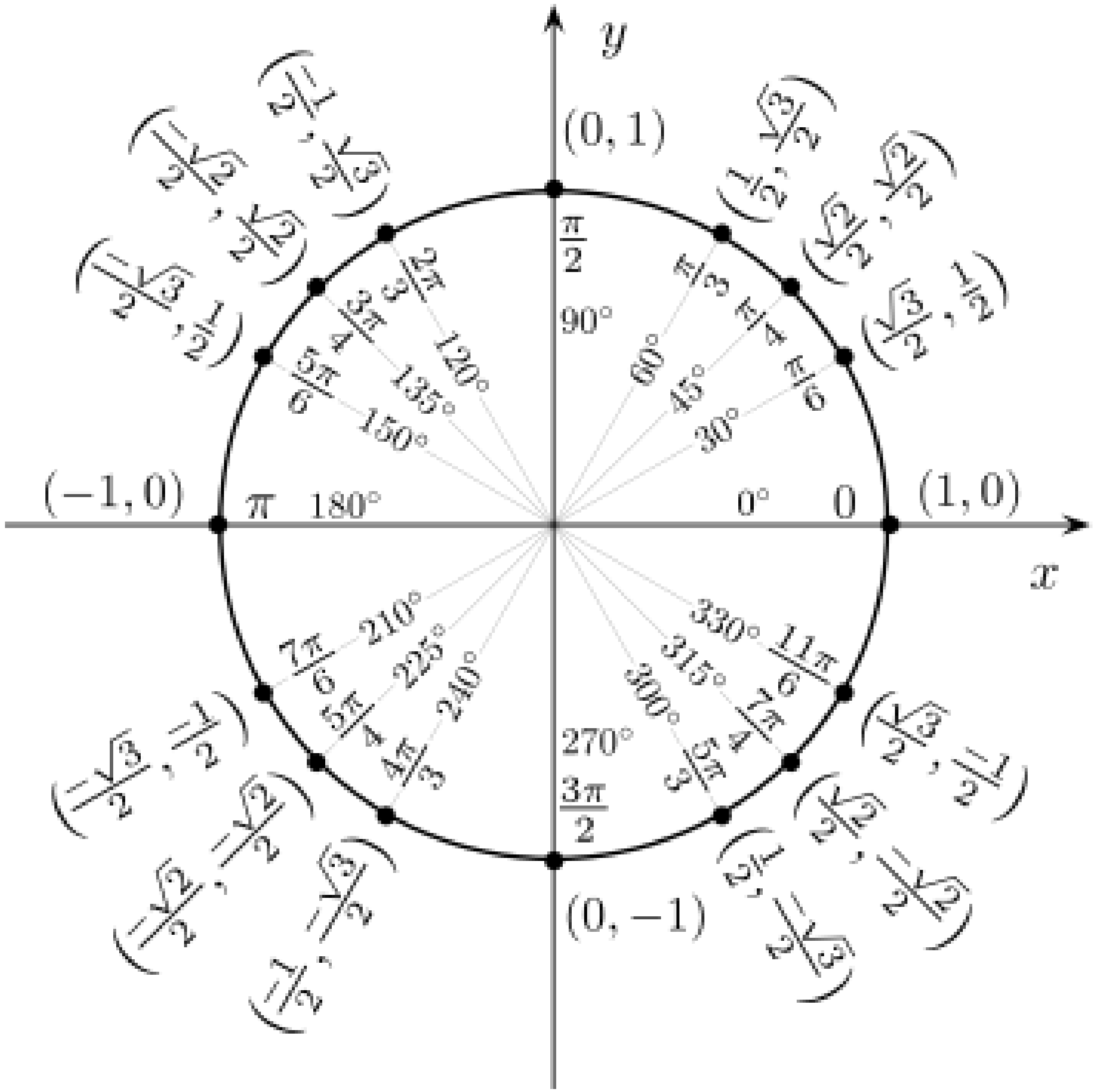
TRIGONOMETRIC FUNCTIONS FOR RIGHT TRIANGLES

The ***sine***, ***cosine***, ***tangent***, ***cosecant***, ***secant***, and ***cotangent*** of a positive acute angle θ can be defined as ratios of the sides of a right triangle. Using the notation from Figure B.6, these definitions take the following form:

$$\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{y}{r}, \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite } \theta} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{x}{r}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to } \theta} = \frac{r}{x} \qquad (6)$$

$$\tan\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{y}{x}, \qquad \cot\theta = \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta} = \frac{x}{y}$$



y
x
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
0°
0
30°
π/6
45°
π/4
60°
π/3
90°
π/2
120°
2π/3
135°
3π/4
150°
5π/6
180°
π
210°
7π/6
225°
5π/4
240°
4π/3
270°
3π/2
300°
5π/3
315°
7π/4
330°
11π/6
(√3/2, 1/2)
(√2/2, √2/2)
(1/2, √3/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
(−1/2, −√3/2)
(1/2, −√3/2)
(√2/2, −√2/2)
(√3/2, −1/2)

INTEGRAL CALCULUS

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

▶ **Example 2** Evaluate $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. There are two possible approaches: we can make the substitution in the indefinite integral (as in Example 1) and then evaluate the definite integral using the x-limits of integration, or we can make the substitution in the definite integral and convert the x-limits to the corresponding θ-limits.

Method 1.

Using the result from Example 1 with the x-limits of integration yields

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}$$

Method 2.

The substitution $x = 2\sin\theta$ can be expressed as $x/2 = \sin\theta$ or $\theta = \sin^{-1}(x/2)$, so the θ-limits that correspond to $x = 1$ and $x = \sqrt{2}$ are

$$x = 1: \quad \theta = \sin^{-1}(1/2) = \pi/6$$

$$x = \sqrt{2}: \quad \theta = \sin^{-1}(\sqrt{2}/2) = \pi/4$$

Thus, from (2) in Example 1 we obtain

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta$$

Convert x-limits to θ-limits.

$$= -\frac{1}{4}\left[\cot\theta\right]_{\pi/6}^{\pi/4} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}$$

INTEGRAL CALCULUS

Table 7.4.1

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

X = a cos Θ

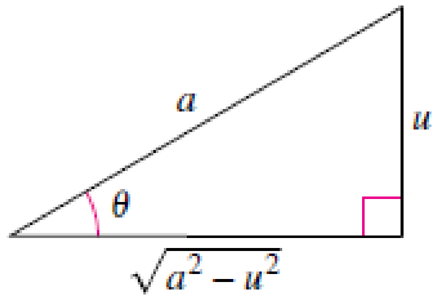
Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a\sin\theta.$$

Then $\sqrt{a^2 - u^2} = a\cos\theta$, where

$$-\pi/2 \le \theta \le \pi/2.$$

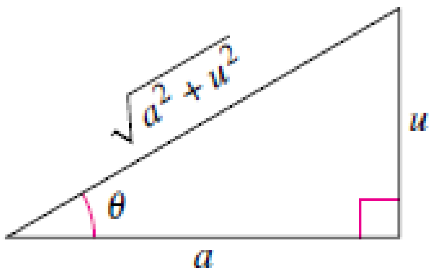


2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a\tan\theta.$$

Then $\sqrt{a^2 + u^2} = a\sec\theta$, where

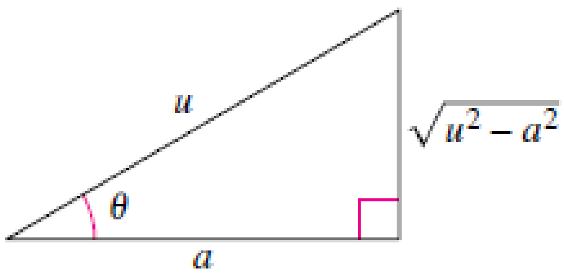
$$-\pi/2 < \theta < \pi/2.$$



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a\sec\theta.$$

Then



$$\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta \text{ for } u > a, \text{ where } 0 \le \theta < \pi/2 \\ -a\tan\theta \text{ for } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$$

▸ **Example 2** Evaluate $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$. $= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x}$

$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x}\Big|_1^{\sqrt{2}}$

$= -\frac{1}{4}\left[\frac{\sqrt{4-(\sqrt{2})^2}}{\sqrt{2}} - \frac{\sqrt{4-(1)^2}}{1}\right]$

$= -\frac{1}{4}\left|\frac{\sqrt{2}}{\sqrt{2}} - \frac{\sqrt{3}}{1}\right|$

$= -\frac{1}{4}(1-\sqrt{3})$ or $\frac{\sqrt{3}-1}{4}$ //

$= -\frac{1}{4}\cot\theta + C$

$= -\frac{1}{4}\cot\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$

$\theta = \sin^{-1}\frac{x}{2}$

$x = \sqrt{2}$

$\theta = \sin^{-1}\frac{\sqrt{2}}{2}$

$= \frac{\pi}{4}$

$x = 1$

$\theta = \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{6}$

$= -\frac{1}{4}\left[\cot\frac{\pi}{4} - \cot\frac{\pi}{6}\right]$

$= -\frac{1}{4}\left[1-\sqrt{3}\right]$

$= \frac{\sqrt{3}-1}{4}$ //

INTEGRAL CALCULUS

TABLE 1.4 Values of sin θ, cos θ, and tan θ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
cos θ	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
tan θ	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

$= \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\sin \frac{1}{2}$

$\cot\frac{\pi}{4} = 1$ — $\cot = \frac{1}{\tan}$ $\quad \frac{1}{1} = 1$

$\cot\frac{\pi}{6} = \sqrt{3}$ — $\tan\frac{\pi}{6}$ $\quad \frac{\sqrt{3}}{3}$

$= \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

▶ Example 5 Evaluate $\int \frac{\sqrt{x^2-25}}{x}\,dx$, assuming that $x \geq 5$.

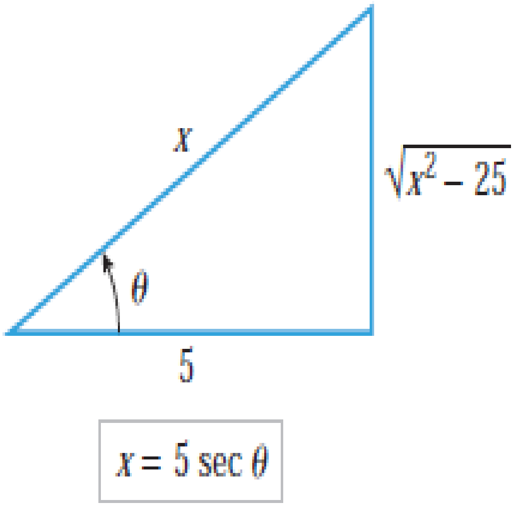
Solution. The integrand involves a radical of the form $\sqrt{x^2-a^2}$ with $a = 5$, so from Table 7.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \leq \theta < \pi/2$$

$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\,d\theta$$

Thus,

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{25\sec^2\theta-25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\ &= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\ &= 5\int \tan^2\theta\,d\theta \qquad \boxed{\tan\theta \geq 0 \text{ since } 0 \leq \theta < \pi/2} \\ &= 5\int(\sec^2\theta-1)\,d\theta = 5\tan\theta - 5\theta + C\end{aligned}$$



▲ Figure 7.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 7.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2 - 25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2 - 25}}{x}\,dx = \sqrt{x^2 - 25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C \blacktriangleleft$$

PARTIAL FRACTIONS

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2(x+1) + 3(x-4)}{(x-4)(x+1)} = \frac{5x-10}{x^2-3x-4} \qquad (1)$$

However, for purposes of integration, the left side of (1) is preferable to the right side since each of the terms is easy to integrate:

$$\int \frac{5x-10}{x^2-3x-4}\,dx = \int \frac{2}{x-4}\,dx + \int \frac{3}{x+1}\,dx = 2\ln|x-4| + 3\ln|x+1| + C$$

Thus, it is desirable to have some method that will enable us to obtain the left side of (1), starting with the right side. To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side. Thus, to find the left side of (1), starting from the right side, we could factor the denominator of the right side and look for constants A and B such that

$$\frac{5x - 10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \tag{2}$$

One way to find the constants A and B is to multiply (2) through by $(x-4)(x+1)$ to clear fractions. This yields

$$5x - 10 = A(x+1) + B(x-4) \tag{3}$$

INTEGRAL CALCULUS

This relationship holds for all x, so it holds in particular if $x = 4$ or $x = -1$. Substituting $x = 4$ in (3) makes the second term on the right drop out and yields the equation $10 = 5A$ or $A = 2$; and substituting $x = -1$ in (3) makes the first term on the right drop out and yields the equation $-15 = -5B$ or $B = 3$. Substituting these values in (2) we obtain

$$\frac{5x - 10}{(x - 4)(x + 1)} = \frac{2}{x - 4} + \frac{3}{x + 1} \tag{4}$$

which agrees with (1).

A second method for finding the constants A and B is to multiply out the right side of (3) and collect like powers of x to obtain

$$5x - 10 = (A + B)x + (A - 4B)$$

Since the polynomials on the two sides are identical, their corresponding coefficients must be the same. Equating the corresponding coefficients on the two sides yields the following system of equations in the unknowns A and B:

$$\begin{aligned} A + B &= 5 \\ A - 4B &= -10 \end{aligned}$$

Solving this system yields $A = 2$ and $B = 3$ as before (verify).

The terms on the right side of (4) are called *partial fractions* of the expression on the left side because they each constitute *part* of that expression. To find those partial fractions we first had to make a guess about their form, and then we had to find the unknown constants. Our next objective is to extend this idea to general rational functions. For this purpose, suppose that $P(x)/Q(x)$ is a *proper rational function*, by which we mean that the degree of the numerator is less than the degree of the denominator. There is a theorem in advanced algebra which states that every proper rational function can be expressed as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where $F_1(x), F_2(x), \ldots, F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

in which the denominators are factors of $Q(x)$. The sum is called the *partial fraction decomposition* of $P(x)/Q(x)$, and the terms are called *partial fractions*. As in our opening example, there are two parts to finding a partial fraction decomposition: determining the exact form of the decomposition and finding the unknown constants.

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = \text{(a polynomial)} + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

GUIDELINES FOR SOLVING THE BASIC EQUATION

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in the first guideline to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.

FINDING THE FORM OF A PARTIAL FRACTION DECOMPOSITION

The first step in finding the form of the partial fraction decomposition of a proper rational function $P(x)/Q(x)$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of *distinct* factors of the form

$$(ax + b)^m \quad \text{and} \quad (ax^2 + bx + c)^m$$

From these factors we can determine the form of the partial fraction decomposition using two rules that we will now discuss.

LINEAR FACTORS

If all of the factors of $Q(x)$ are linear, then the partial fraction decomposition of $P(x)/Q(x)$ can be determined by using the following rule:

LINEAR FACTOR RULE For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 1 Evaluate $\int \frac{dx}{x^2 + x - 2}$.

Solution. The integrand is a proper rational function that can be written as

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)}$$

The factors $x - 1$ and $x + 2$ are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \tag{5}$$

where A and B are constants to be determined. Multiplying this expression through by $(x-1)(x+2)$ yields

$$1 = A(x+2) + B(x-1) \tag{6}$$

Setting $x = 1$ makes the second term in (6) drop out and yields $1 = 3A$ or $A = \frac{1}{3}$; and setting $x = -2$ makes the first term in (6) drop out and yields $1 = -3B$ or $B = -\frac{1}{3}$. Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

The integration can now be completed as follows:

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C \blacktriangleleft$$

Example 2 Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$.

Solution. The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Although x^2 is a quadratic factor, it is *not* irreducible since $x^2 = xx$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

and the factor $x - 2$ introduces one term (since $m = 1$) of the form

$$\frac{C}{x-2}$$

so the partial fraction decomposition is

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \tag{7}$$

Multiplying by $x^2(x-2)$ yields

$$2x+4 = Ax(x-2) + B(x-2) + Cx^2 \tag{8}$$

which, after multiplying out and collecting like powers of x, becomes

$$2x+4 = (A+C)x^2 + (-2A+B)x - 2B \tag{9}$$

Setting $x = 0$ in (8) makes the first and third terms drop out and yields $B = -2$, and setting $x = 2$ in (8) makes the first and second terms drop out and yields $C = 2$ (verify). However, there is no substitution in (8) that produces A directly, so we look to Equation (9) to find this value. This can be done by equating the coefficients of x^2 on the two sides to obtain

$$A + C = 0 \quad \text{or} \quad A = -C = -2$$

Substituting the values $A = -2$, $B = -2$, and $C = 2$ in (7) yields the partial fraction decomposition

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

Thus,

$$\int \frac{2x+4}{x^2(x-2)}\,dx = -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C$$

INTEGRAL CALCULUS

Example 2 Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$.

$$= \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$x^2(x-2)\left(\frac{2x+4}{x^2(x-2)}\right) = \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}\right)\left(x^2(x-2)\right)$$

$$2x+4 = Ax(x-2) + B(x-2) + C(x^2) \quad (1)$$

$$2x+4 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$2x+4 = (A+C)x^2 + (-2A+B)x - 2B \quad (2)$$

$x=0$: $2(0)+4 = A(0) + B(0-2) + C(0)$

$$4 = -2B$$

$$B = -2$$

$x=2$: $2(2)+4 = A(0) + B(0) + C\,4$

$$8 = 4C \qquad C = 2$$

$$A + C = 0$$

$$A = -C$$

$$A = -2$$

$A = -2 \quad B = -2$

$C = 2$

$$= -\frac{2}{x} - \frac{2}{x^2} + \frac{2}{x-2}$$

$$= \int\left(\frac{-2}{x} - \frac{2}{x^2} + \frac{2}{x-2}\right)dx$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

or

$$= 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C$$

EX 3

$$\int \frac{dx}{x^2 - 3x - 4}$$

$$\frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}; \qquad A = \frac{1}{5},\ B = -\frac{1}{5},$$

$$\frac{1}{5}\int \frac{1}{x-4}dx - \frac{1}{5}\int \frac{1}{x+1}dx = \frac{1}{5}\ln|x-4| - \frac{1}{5}\ln|x+1| + C$$

$$= \frac{1}{5}\ln\left|\frac{x-4}{x+1}\right| + C.$$

examples

examples

Partial fraction using method 1

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

$$(x-1)(x+2)\,\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-1)$$

let $x = -2$

$$1 = A(-2+2) + B(-2-1)$$

$$1 = 0 - 3B$$

$$-1 = 3B \qquad B = -\frac{1}{3}$$

$(x-1)(x+2)$

$$1 = A(x+2) + B(x-1)$$

let $x = 1$

$$1 = A(1+2) + B(1-1)$$

$$1 = 3A$$

$$A = \frac{1}{3}$$

$$\frac{1}{x^2+x-2} = \frac{1/3}{x-1} + \frac{-1/2}{x+2}$$

$$= \frac{1}{3(x-1)} - \frac{1}{2(x+2)}$$

method 2 cover up method

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

A

$x-1=0$

$x=1$

$\frac{1}{x+2}$

$A = \frac{1}{1+2} = \frac{1}{3}$

B

$x+2=0$

$x=-2$

$B = \frac{1}{x-1} = \frac{1}{-2-1} = -\frac{1}{3}$

$$\int \frac{dx}{(x-1)(x+2)} = \int \frac{1}{3} \frac{dx}{x-1} - \int \frac{1}{3} \frac{dx}{x+2}$$

$$= \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{dx}{x+2}$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| + c$$

or

$$\boxed{= \frac{1}{3} \ln\left|\frac{x-1}{x+2}\right| + c}$$

2) $\displaystyle\int \frac{dx}{(x^2-3x-4)} = \frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$

Using Cover up

A: $x-4=0$, $x=4$ | $A = \frac{1}{x+1} = \frac{1}{4+1} = \frac{1}{5}$

B: $x+1=0$, $x=-1$ | $B = \frac{1}{x-4} = \frac{1}{-1-4} = -\frac{1}{5}$

$$\int \frac{1/5\,dx}{x-4} + \frac{-1/5\,dx}{x+1} = \frac{1}{5}\int \frac{dx}{x-4} - \frac{1}{5}\int \frac{dx}{x+1}$$

$$= \frac{1}{5}\ln|x-4| - \frac{1}{5}\ln|x+1| + c$$

Or by laws of log

$$\boxed{= \frac{1}{5}\ln\left|\frac{x-1}{x+1}\right| + c}$$

$$\int \frac{2x+4}{x^3-2x^2}dx = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$x^2(x-2$ $\quad \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \quad x^2(x-2)$

$2x+4 = Ax(x-2) + B(x-2) + Cx^2$ ①

$2x+4 = Ax^2 - 2Ax + Bx - 2B + Cx^2$

$2x+4 = Ax^2 + Cx^2 - 2Ax + Bx$

$2x+4 = (A+C)x^2 + (-2A+B)x$

$A+C = 0$ ② $\quad -2A+B = 2$ ③

at $x=0$

$2(0)+4 = A(0)(0-2) + B(0-2) + C(0)^2$

$4 = -2B$

$B = -2$

Using eq ③

$-2A+B = 2$

$-2A-2 = 2$

$-2A = 4$

$A = -2$

using eq 2

$A+C = 0$

$-2+C = 0$

$C = 2$

$A = -2 \quad B = -2 \quad C = 2$

$$\frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2} = \int \frac{-2dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$= -2\ln|x| + \frac{2}{x} + 2\ln|x-1| + c$

or $2\ln|x-1| - 2\ln x + \frac{2}{x} + c$

or $2\ln\left|\frac{x-1}{x}\right| + \frac{2}{x} + C$

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

C

$x-2=0$

$x=2$

$$\left.\frac{2x+4}{x^2}\right| = \frac{8}{4} = 2 \qquad C = 2$$

B

$x=0$

$$\left.\frac{d}{dx}\frac{2x+4}{x-2}\right| = \frac{4}{-2} = \quad B = -2$$

A

$x=0$

$$d\left(\frac{2x+4}{x-2}\right) = \frac{[(x-2)(2)] - [(2x+4)(1)]}{(x-2)^2} = \frac{(2x-4)-(2x+4)}{(x-2)^2}$$

$$= \frac{2x-4-2x-4}{(x-2)^2} = \frac{-8}{(x-2)^2} = \frac{-8}{4} = -2$$

$A = -2$

QUADRATIC FACTORS

If some of the factors of $Q(x)$ are irreducible quadratics, then the contribution of those factors to the partial fraction decomposition of $P(x)/Q(x)$ can be determined from the following rule:

QUADRATIC FACTOR RULE For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

▶ Example 3 Evaluate $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\, dx$.

Quadratic Factor

$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$$

$$\int \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

$$\left[\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}\right] x(x-1)(x^2+4) \quad (1)$$

$$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

$$(x+D)\ x^2-x$$

$$Cx^3 - Cx^2 + Dx^2 - Dx$$

$$2x^3-4x-8 \equiv A(x^3+4x-x^2-4) + B(x^3+4x) + Cx^3 - Cx^2 + Dx^2 - Dx$$

$$= Ax^3 + 4Ax - Ax^2 - 4A + Bx^3 + 4Bx + Cx^3 - Cx^2 + Dx^2 - Dx$$

$$= Ax^3 + Bx^3 + Cx^3 - Ax^2 - Cx^2 + Dx^2 + 4Ax + 4Bx - Dx - 4A$$

$$= (A+B+C)x^3 + (-A-C+D)x^2 + (4A+4B-D)x - 4A$$

$-8 = -4A$	$A+B+C = 2$	$-A-C+D = 0$	$4A+4B-D = -4$
$A = 2$	$2+B+C = 2$	$-2-C+D = 0$	$8+4B-D = -4$
	$B+C = 0$	$-C+D = 2$	$4B-D = -12$
	$-2+C = 0$	$-2+D = 2$	$-8-D = -12$
	$C = 2$	$D = 4$	

$x = +1$

$2-4-8 = B(5)$

$-10 = 5B$

$B = -2$

11

$$\int \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

$A=2$

$B=-2$

$C=2$

$D=4$

$$= \frac{2}{x} + \frac{-2}{x-1} + \frac{2x+4}{x^2+4}$$

$$\int\left(\frac{2}{x} + \frac{-2}{x-1} + \frac{2x+4}{x^2+4}\right)dx$$

$$= 2\int\frac{dx}{x} - 2\int\frac{dx}{x-1} + \int\frac{2x}{x^2+4}dx + \int\frac{4}{x^2+4}$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\tan^{-1}\frac{x}{2} + C$$

$$= 2\ln\left|\frac{x}{x-1}\right| + \ln|x^2+4| + 2\tan^{-1}\frac{x}{2} + C$$

$$\int \frac{x^2+x-2}{(3x^3-x^2)+(3x-1)}\,dx$$

$$\int \frac{x^2+x-2}{x^2(3x-1)+(3x-1)}\,dx$$

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$\left[\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}\right](3x-1)(x^2+1)$$

$$x^2+x-2 = A(x^2+1)+(Bx+C)(3x-1)$$

$$x^2+x-2 = Ax^2+A+3Bx^2-Bx+3Cx-C$$

$$x^2+x-2 = (Ax^2+3Bx^2)-Bx+3Cx+A-C$$

$$= (A+3B)x^2+(-B+3C)x+A-C$$

$A+3B=1$ ① $\quad -B+3C=1$ ② $\quad A-C=-2$ ③

$A=-2+C$ ④

$-2+C+3B=1$

$3B+C=3$ ⑤

$3B+C=3$

$(3)(-B+3C=1)$

$$\begin{aligned} 3B+C&=3\\ -3B+9C&=3\\ \hline 10C&=6 \end{aligned}$$

$C=\frac{6}{10}=\frac{3}{5}$

$A=-2+C$

$C=\frac{3}{5}$

$A=-2+\frac{3}{5}$

$A=\frac{-10+3}{5}=\frac{-7}{5}$

$-B+3C=1$

$-B+3\left(\frac{3}{5}\right)=1$

$-B+\frac{9}{5}=1$

$-B=1-\frac{9}{5}$

$-B=\frac{5-9}{5}$

$B=\frac{9}{5}$

$$\frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{-7/5}{3x-1} + \frac{\frac{9}{5}x + \frac{3}{5}}{x^2+1}$$

$$-\frac{7}{5}\int \frac{dx}{3x-1} + \frac{9}{5}\int \frac{x\,dx}{x^2+1} + \frac{3}{5}\int \frac{dx}{x^2+1}$$

$$= -\frac{7}{5} \ln|3x-1| + \frac{9}{5}\left[\frac{1}{2}\ln|x^2+1| + \frac{3}{5} \tan^{-1}x + C\right.$$

$$\boxed{= -\frac{7}{5}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}(x) + C}$$

Solution. The denominator in the integrand can be factored by grouping:

$$3x^3 - x^2 + 3x - 1 = x^2(3x-1) + (3x-1) = (3x-1)(x^2+1)$$

By the linear factor rule, the factor $3x - 1$ introduces one term, namely,

$$\frac{A}{3x-1}$$

and by the quadratic factor rule, the factor $x^2 + 1$ introduces one term, namely,

$$\frac{Bx+C}{x^2+1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} \qquad (10)$$

Multiplying by $(3x-1)(x^2+1)$ yields

$$x^2 + x - 2 = A(x^2+1) + (Bx+C)(3x-1) \qquad (11)$$

We could find A by substituting $x = \frac{1}{3}$ to make the last term drop out, and then find the rest of the constants by equating corresponding coefficients. However, in this case it is just as easy to find *all* of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the right side of (11) and collect like terms:

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A + 3B \qquad &= 1 \\ - B + 3C &= 1 \\ A \qquad - C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

INTEGRAL CALCULUS

$A + 3B = 1$ (1)

$-B + 3C = 1$ (2)

$A - C = -2$ (3) $A = -2 + C$

$-2 + C + 3B = 1$

$3B + C = 3$ (1)

$(3)\ -B + 3C = 1$ (2)

$3B + C = 3$

$-3B + 9C = 3$

$0 \quad 10C = 6$

$C = \frac{6}{10}$ or $\frac{3}{5}$ $C = \frac{3}{5}$

$A = -2 + \frac{3}{5}$

$A = -\frac{7}{5}$

$-B + 3\left(\frac{3}{5}\right) = 1$

$-B + \frac{9}{5} = 1$

$B = \frac{4}{5}$

Thus, (10) becomes

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}$$

and

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx = -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1}$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C \blacktriangleleft$$

Example 4 Evaluate $\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} dx$.

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A + 3B \qquad &= 1 \\ -B + 3C &= 1 \\ A \qquad - C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

Solution. Observe that the integrand is a proper rational function since the numerator has degree 4 and the denominator has degree 5. Thus, the method of partial fractions is applicable. By the linear factor rule, the factor $x + 2$ introduces the single term

$$\frac{A}{x+2}$$

and by the quadratic factor rule, the factor $(x^2 + 3)^2$ introduces two terms (since $m = 2$):

$$\frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2}$$

Thus, the partial fraction decomposition of the integrand is

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2} \qquad (12)$$

Multiplying by $(x + 2)(x^2 + 3)^2$ yields

$$3x^4 + 4x^3 + 16x^2 + 20x + 9$$
$$= A(x^2 + 3)^2 + (Bx + C)(x^2 + 3)(x + 2) + (Dx + E)(x + 2) \qquad (13)$$

which, after multiplying out and collecting like powers of x, becomes

$$\begin{aligned} 3x^4 + 4x^3 + 16x^2 + 20x + 9 \\ &= (A + B)x^4 + (2B + C)x^3 + (6A + 3B + 2C + D)x^2 \\ &\quad + (6B + 3C + 2D + E)x + (9A + 6C + 2E) \end{aligned} \tag{14}$$

Equating corresponding coefficients in (14) yields the following system of five linear equations in five unknowns:

$$\begin{aligned} A + B &= 3 \\ 2B + C &= 4 \\ 6A + 3B + 2C + D &= 16 \\ 6B + 3C + 2D + E &= 20 \\ 9A + 6C + 2E &= 9 \end{aligned} \tag{15}$$

in (13), which yields A = 1. Substituting this known value of A in (15) yields the simpler system

$$\begin{aligned} B &= 2 \\ 2B + C &= 4 \\ 3B + 2C + D &= 10 \\ 6B + 3C + 2D + E &= 20 \\ 6C + 2E &= 0 \end{aligned} \tag{16}$$

This system can be solved by starting at the top and working down, first substituting $B = 2$ in the second equation to get $C = 0$, then substituting the known values of B and C in the third equation to get $D = 4$, and so forth. This yields

$$A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0$$

Thus, (12) becomes

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

and so

$$\begin{aligned}\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\,dx &= \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\,dx + 4\int \frac{x}{(x^2+3)^2}\,dx \\ &= \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C\end{aligned}$$