CALCULUS 2

INTEGRATION TECHNIQUES

Integration by parts

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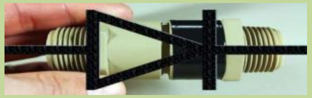
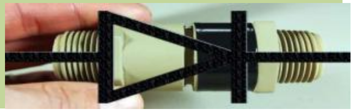


OBJECTIVES



- to evaluate integrals using reduction formulas

TRANSFORMATIONS of TRIGONOMETRIC FUNCTIONS

The transformation of the trigonometric functions are divided into two major parts, they are the following:

Part 1: Powers of Sine and Cosine

Part 2: Powers of Tangent and Secant and
Powers of Cotangent and Cosecant

Part 1: Powers of Sine and Cosine

$Case\,1:$ Consider the integrand

$$\int \sin^{n} u\, du \;\text{ or }\; \int \cos^{n} u\, du \quad \text{where n is any positive odd integer}$$

use the identity : $\sin^{n} u + \cos^{n} u = 1$

$Case\,2:$ Consider the integrand

$$\int \sin^{n} u\, du \;\text{ or }\; \int \cos^{n} u\, du \quad \text{where n is any positive even integer}$$

use the identity : $\sin^{n} u = \frac{1}{2}\left(1 - \cos 2u\right)$

$$\cos^{n} u = \frac{1}{2}\left(1 + \cos 2u\right)$$

INTEGRAL CALCULUS

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \qquad (1)$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \qquad (2)$$

In the case where $n = 2$, these formulas yield

$$\int \sin^2 x\,dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x - \tfrac{1}{2}\sin x \cos x + C \qquad (3)$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x \cos x + C \qquad (4)$$

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Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x) \tag{5–6}$$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2\sin^2 x \qquad \text{and} \qquad \cos 2x = 2\cos^2 x - 1$$

These identities yield

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{7}$$

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \tag{8}$$

Observe that the antiderivatives in Formulas (3) and (4) involve both sines and cosines, whereas those in (7) and (8) involve sines alone. However, the apparent discrepancy is easy to resolve by using the identity

$$\sin 2x = 2 \sin x \cos x$$

to rewrite (7) and (8) in forms (3) and (4), or conversely.

In the case where $n = 3$, the reduction formulas for integrating $\sin^3 x$ and $\cos^3 x$ yield

$$\int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\,dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C \qquad (9)$$

$$\int \cos^3 x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\int \cos x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C \qquad (10)$$

$$\int \sin^3 x \, dx = \tfrac{1}{3}\cos^3 x - \cos x + C \tag{11}$$

$$\int \cos^3 x \, dx = \sin x - \tfrac{1}{3}\sin^3 x + C \tag{12}$$

$$\int \sin^4 x \, dx = \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \tag{13}$$

$$\int \cos^4 x \, dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \tag{14}$$

$$\int \sin^4 x\, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C \qquad (13)$$

$\int \sin^4 x\, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

Case 2 : Consider the integrand

$\int \sin^n u\, du$ or $\int \cos^n u\, du$ where n is any positive even integer

use the identity : $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

$\int \sin^2 x\, dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C$

$\int \sin^4 x\, dx$

$\int (\sin^2 x)^2\, dx = \int \left[\frac{1}{2}(1 - \cos 2x)\right]^2 dx$

$= \int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$= \int \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right] dx$

$= \int \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right] dx$

$= \int \left(\frac{1}{4} - \frac{2}{4}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\right) dx$

$= \int \left[\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right] dx$

$= \int \frac{3}{8}\, dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{8}\int \cos 4x\, dx$

$= \frac{3}{8}x - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + \frac{1}{8}\left(\frac{1}{4}\sin 4x\right) + C$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

$\cos^2 u$

$u = 2x$

$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

$= \frac{1}{2}(1 + \cos 2(2x))$

$= \frac{1}{2}(1 + \cos 4x)$

$$\int \cos^4 x \, dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \qquad (14)$$

$$\int \cos^4 x \, dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

Case 2 : Consider the integrand

$\int \sin^n u \, du$ or $\int \cos^n u \, du$ where n is any positive even integer

use the identity : $\sin^n u = \frac{1}{2}(1 - \cos 2u)$

$\cos^n u = \frac{1}{2}(1 + \cos 2u)$

$$\int (\cos^2 x)^2 dx$$

$$= \int \left(\tfrac{1}{2}(1 + \cos 2x)\right)^2 dx$$

$$= \int \left[\tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x\right] dx$$

$\cos^2 2x = \frac{1}{2}(1 + \cos 2u)$

$u = 2x$

$= \frac{1}{2}(1 + \cos 2(2x))$

$= \frac{1}{2}(1 + \cos 4x)$

$$= \int \left[\tfrac{1}{4}(1 + 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x\right] dx$$

$$= \int \tfrac{1}{4}(1 + 2\cos 2x + \tfrac{1}{2} + \tfrac{1}{2}\cos 4x] \, dx$$

$$= \int \left(\tfrac{1}{4} + \tfrac{2}{4}\cos 2x + \tfrac{1}{8} + \tfrac{1}{8}\cos 4x\right) dx$$

$$= \int \left(\tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x\right) dx$$

$$= \int \tfrac{3}{8} dx + \int \tfrac{1}{2}\cos 2x \, dx + \int \tfrac{1}{8}\cos 4x \, dx$$

$$= \tfrac{3}{8}x + \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2x\right) + \tfrac{1}{8}\left(\tfrac{1}{4}\sin 4x\right) + C \rightarrow \boxed{\tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C}$$

INTEGRATING PRODUCTS OF SINES AND COSINES

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by one of the three procedures stated in Table 7.3.1, depending on whether m and n are odd or even.

Example 2 Evaluate

(a) $\int \sin^4 x \cos^5 x \, dx$ (b) $\int \sin^4 x \cos^4 x \, dx$

INTEGRAL CALCULUS

Table 7.3.1

INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Solution (a). Since $n = 5$ is odd, we will follow the first procedure in Table 7.3.1:

$$\begin{aligned}
\int \sin^4 x \cos^5 x\, dx &= \int \sin^4 x \cos^4 x \cos x\, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x\, dx \\
&= \int u^4 (1 - u^2)^2\, du \\
&= \int (u^4 - 2u^6 + u^8)\, du \\
&= \tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9 + C \\
&= \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C
\end{aligned}$$

$\int \sin^4 x \cos^5 x \, dx$

$\int \sin^m x \cos^n x \, dx$

$m = 4$ even

$n = 5$ odd

case 1

$\int \sin^4 x \cos^4 x \ \cos x \, dx$ use $\cos^2 x = 1 - \sin^2 x$

$\int \sin^4 x \, (\cos^2 x)^2 \cos x \, dx$

$\int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$

let $u = \sin x$ $\quad du = \cos x \, dx$

$\int u^4 (1 - u^2)^2 du$

$\int (u^4 (1 - 2u^2 + u^4) \, du$

$\int (u^4 - 2u^6 + u^8) \, du$

$\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + c$

$\frac{(\sin x)^5}{5} - \frac{2(\sin x)^7}{7} + \frac{(\sin x)^9}{9} + C$

$$\boxed{\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + c}$$

Solution (b). Since $m = n = 4$, both exponents are even, so we will follow the third procedure in Table 7.3.1:

$$\int \sin^4 x \cos^4 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left(\tfrac{1}{2}[1 - \cos 2x]\right)^2 \left(\tfrac{1}{2}[1 + \cos 2x]\right)^2 \, dx$$

$$= \tfrac{1}{16} \int (1 - \cos^2 2x)^2 \, dx$$

$$= \tfrac{1}{16} \int \sin^4 2x \, dx$$

Note that this can be obtained more directly from the original integral using the identity $\sin x \cos x = \frac{1}{2} \sin 2x$.

$$= \tfrac{1}{32} \int \sin^4 u \, du$$

$u = 2x$
$du = 2\,dx$ or $dx = \frac{1}{2}\,du$

$$= \tfrac{1}{32} \left(\tfrac{3}{8}u - \tfrac{1}{4} \sin 2u + \tfrac{1}{32} \sin 4u\right) + C$$

Formula (13)

$$= \tfrac{3}{128}x - \tfrac{1}{128} \sin 4x + \tfrac{1}{1024} \sin 8x + C \blacktriangleleft$$

$\int \sin^4 x \cos^4 x \, dx$

$m=4 \quad n=4$ both even / apply case 3

$= \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$

$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$

$= \int \frac{1}{4}(1-\cos^2 2x)\, \frac{1}{4}(1+\cos^2 2x)\, dx$

$= \frac{1}{16}\int (1-\cos^4 2x)\, dx$

$= \frac{1}{16}\int \frac{1}{2}\sin^4 2x \, dx$

$= \frac{1}{16}\int \sin^4 2x \, dx \qquad u = 2x, \quad du = 2\,dx, \quad \frac{du}{2} = dx$

$= \frac{1}{16}\int \sin^4 u \, \frac{du}{2}$

$= \frac{1}{32}\int \sin^4 u \, du$

using formula 13

$\int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

$= \frac{1}{32}\left[\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right] + c$

$= \frac{1}{32}\left[\frac{3}{8}(2x) - \frac{1}{4}\sin 2(2x) + \frac{1}{32}\sin 4(2x)\right]$

$= \frac{1}{32}\left[\frac{6}{8}x - \frac{1}{4}\sin 4x + \frac{1}{32}\sin 8x\right] + c$

$= \frac{6}{256}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$

$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$

Ex. for case 2

$\int \sin^3 x \cos^4 x \, dx$

$m = 3$ odd use case 2

use identities

$\sin^2 x = 1 - \cos^2 x$

$= \int \sin^2 x \cos^4 x \sin x \, dx$

$= \int (1 - \cos^2 x) \cos^4 x \sin x \, dx$

let $u = \cos x$ $du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \int (1 - u^2) u^4 \, -du$

$= -\int (1 - u^2) u^4 \, du$

$= -\int (u^4 - u^6) \, du$

$= -\left[\frac{u^5}{5} - \frac{u^7}{7} \right] + c$

$= -\frac{1}{5} u^5 + \frac{1}{7} u^7 + c$

$= -\frac{1}{5} (\cos x)^5 + \frac{1}{7} (\cos x)^7 + c$

$$= -\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C$$

Integrals of the form

$$\int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx \tag{15}$$

can be found by using the trigonometric identities

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \tag{16}$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{17}$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{18}$$

to express the integrand as a sum or difference of sines and cosines.

Example 3 Evaluate $\int \sin 7x \cos 3x \, dx$.

$\int \sin 7x \cos 3x \, dx$

$a = 7 \quad B = 3$

use formula 16

$\sin a \cos B = \frac{1}{2}[\sin(a-B) + \sin(a+B)]$

$= \int \frac{1}{2} [\sin(7-3)x + \sin(7+3)x] dx$

$= \frac{1}{2} \int (\sin 4x + \sin 10x) dx$

$= \frac{1}{2} [-\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x] + C$

$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$

13. $\int \sin 2x \cos 3x \, dx$

solutions

13. $\int \sin 2x \cos 3x \, dx = \frac{1}{2} \int (\sin 5x - \sin x) dx = -\frac{1}{10} \cos 5x + \frac{1}{2} \cos x + C.$

INTEGRAL CALCULUS

Case 3: If the intergrand contains the product of sine and cosine functions of the form $\int \sin^{n} u \cdot \cos^{m} u \, du$ where at least one of the m or n is a positive odd integer, employ the same technique as that of Case 1.

Case 4: If the intergrand contains the product of sine and cosine functions of the form $\int \sin^{n} u \cdot \cos^{m} u \, du$ where both m or n is a positive even integer, employ the same technique as that of Case 2.

Case 5: If the intergrand has any of the following form:

$$\int \sin au \cdot \cos bu \, \text{du}; \quad \int \cos au \cdot \cos bu \, \text{du}; \quad \int \sin au \cdot \sin bu \, \text{du}$$

Use the following transformations:

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left[-\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$where\ \alpha = \text{au and } \beta = \text{bu}$$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x\, dx$$

can be evaluated by one of the three procedures stated in Table 7.3.2, depending on whether m and n are odd or even.

Table 7.3.2

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Example 4 Evaluate

(a) $\int \tan^2 x \sec^4 x \, dx$

a) $\int \tan^2 x \sec^4 x \, dx$

$m=2 \quad n=4$ even

use case 1

$\sec^2 x = \tan^2 x + 1$

$\int \tan^2 x \sec^2 x \sec^2 x \, dx$

$\int \tan^2 x \, (\tan^2 x + 1) \sec^2 x \, dx$

$u = \tan x \qquad du = \sec^2 x \, dx$

$= \int u^2 (u^2 + 1) \, du$

$= \int (u^4 + u^2) \, du$

$= \frac{u^5}{5} + \frac{u^3}{3} + c$

$= \frac{1}{5}(\tan x)^5 + \frac{1}{3}(\tan x)^3 + c$

$$\boxed{= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c}$$

ASSIGNMENT

1. $\int \sin x \cos(x/2)\, dx$

2. $\int \tan^3 x \sec^3 x\, dx$

3. $\int \tan^2 x \sec x\, dx$