CALCULUS 2
LESSON 9

INTEGRATION TECHNIQUES

Integration by parts

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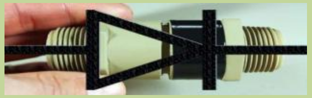
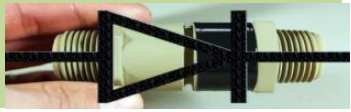


OBJECTIVES



- to evaluate integrals using integration by parts
- integrate functions using repeated integration by parts
- Integrate functions using tabular form

Integration by Parts:

The most useful among the techniques of integration is the integration by parts.

It is derived from the differentials of the product of two factors. If u and v are both differentiable functions of x, then

$$d(uv) = udv + vdu$$

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$$d(uv) = udv + vdu$$

By transposition,

$$udv = d(uv) - vdu$$

Integrating both sides of the equation, we have

$$\int udv = uv - \int vdu$$

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The integral $\int udv$ is expressed in terms of another

integral $\int vdu$ which must be simpler than the given integral,

and is easier to evaluate.

Thus, given an integrand, a factor may be set as u, which is differentiable, and the other part as dv where its integral must exist. The process can be used repeatedly.

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The technique is used in integrating odd powers of :

- odd powers secant, cosecant, hyperbolic secant and hyperbolic cosecant like ,

$$\int \sec^3 4x\,dx \qquad \int x\,\text{csch}^5 x^2\,dx$$

- inverses of trigonometric and hyperbolic functions like,

$$\int \sin^{-1} 2x\,dx \qquad \int x\cosh^{-1} 3x\,dx$$

- products of transcendental /algebraic functions like

$$\int x^2 \sin 4x\,dx \qquad \int e^{2x}\cos x\,dx$$

Example 1 Use integration by parts to evaluate $\int x \cos x \, dx$.

Solution. We will apply Formula (3). The first step is to make a choice for u and dv to put the given integral in the form $\int u \, dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x \, dx$$

The second step is to compute *du* from *u* and *v* from *dv*. This yields

The third step is to apply Formula (3). This yields

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = \underbrace{x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \underbrace{dx}_{du}$$

$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C$$

Find $\int x \sin x \, dx$.

We have three choices: (*a*) $u = x \sin x$, $dv = dx$; (*b*) $u = \sin x$, $dv = x\,dx$; (*c*) $u = x$, $dv = \sin x \, dx$.

(*a*) Let $u = x \sin x$, $dv = dx$. Then $du = (\sin x + x \cos x)\,dx$, $v = x$, and

$$\int x \sin x \, dx = x \cdot x \sin x - \int x(\sin x + x \cos x)\,dx$$

The resulting integral is not as simple as the original, and this choice is discarded.

(*b*) Let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x \, dx$, $v = \frac{1}{2}x^2$, and

$$\int x \sin x \, dx = \tfrac{1}{2}x^2 \sin x - \int \tfrac{1}{2}x^2 \cos x \, dx$$

The resulting integral is not as simple as the original, and this choice too is discarded.

(*c*) Let $u = x$, $dv = \sin x \, dx$. Then $du = dx$, $v = -\cos x$, and

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \sin x + C$$

The integral of $\ln(x)$ can be calculated using integration by parts. Let's denote $u = \ln(x)$ and $dv = dx$. Then, we have $du = \frac{1}{x}dx$ and $v = x$.

Using the integration by parts formula:

$$\int \ln(x)\, dx = uv - \int v\, du$$

$$= x \ln(x) - \int x \cdot \frac{1}{x}\, dx$$

$$= x \ln(x) - \int 1\, dx$$

$$= x \ln(x) - x + C$$

GUIDELINES FOR INTEGRATION BY PARTS

- The main goal in integration by parts is to choose *u* and *dv* to obtain a new integral that is easier to evaluate than the original.
- In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice.
- A strategy that often works is to choose *u* and *dv* so that *u* becomes "simpler" when differentiated, while leaving a *dv* that can be readily integrated to obtain *v*.

There is another useful strategy for choosing *u* and *dv* that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

EXAMPLE 1 Integration by Parts

Find $\int xe^x \, dx$.

Solution To apply integration by parts, you need to write the integral in the form $\int u \, dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x \, dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x \, dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\underbrace{(xe^x \, dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines on the preceding page suggest the first option because the derivative of $u = x$ is simpler than x, and $dv = e^x \, dx$ is the most complicated portion of the integrand that fits a basic integration formula.

$$dv = e^x \, dx \quad \Longrightarrow \quad v = \int dv = \int e^x \, dx = e^x$$

$$u = x \quad \Longrightarrow \quad du = dx$$

Now, integration by parts produces

$$\int u \, dv = uv - \int v \, du \qquad \text{Integration by parts formula}$$

$$\int xe^x \, dx = xe^x - \int e^x \, dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

Example 2 Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$

▶ **Example 3** Evaluate $\int \ln x \, dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x \, dx$. But with this choice finding v is equivalent to evaluating $\int \ln x \, dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x \, dx = \int u \, dv = uv - \int v \, du = x \ln x - \int dx = x \ln x - x + C \ ◀$$

EXAMPLE 2 Integration by Parts

Find $\int x^2 \ln x \, dx$.

Solution In this case, x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should let $dv = x^2\,dx$.

$$dv = x^2\,dx \quad \Rightarrow \quad v = \int x^2\,dx = \frac{x^3}{3}$$

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$$

Integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx \qquad \text{Substitute.}$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \qquad \text{Simplify.}$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \text{Integrate.}$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9} + C\right] = \frac{x^3}{3}\left(\frac{1}{x}\right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$

EXAMPLE 1: Find $\int x^3 e^{x^2}\,dx$.

Take $u = x^2$ and $dv = e^{x^2} x\,dx$; then $du = 2x\,dx$ and $v = \frac{1}{2} e^{x^2}$. Now by (*31.1*),

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2} x^2 e^{x^2} - \int x\,e^{x^2}\,dx = \tfrac{1}{2} x^2 e^{x^2} - \tfrac{1}{2} e^{x^2} + C$$

2. Find $\int xe^x\,dx$.

Let $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

3. Find $\int x^2 \ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

4. Find $\int x\sqrt{1+x}\,dx$.

Let $u = x$, $dv = \sqrt{1+x}\,dx$. Then $du = dx$, $v = \frac{2}{3}(1+x)^{3/2}$, and

$$\int x\sqrt{1+x}\,dx = \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2}\,dx = \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C$$

5. Find $\int \arcsin x\,dx$.

Let $u = \arcsin x$, $dv = dx$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$, and

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C$$

6. Find $\int \sin^2 x \, dx$.

Let $u = \sin x$, $dv = \sin x \, dx$. Then $du = \cos x \, dx$, $v = -\cos x$, and

$$\int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx = -\sin x \cos x + \int (1 - \sin^2 x) \, dx$$

$$= -\tfrac{1}{2} \sin 2x + \int dx - \int \sin^2 x \, dx$$

Hence $\quad 2\int \sin^2 x \, dx = -\tfrac{1}{2} \sin 2x + x + C' \quad$ and $\quad \int \sin^2 x \, dx = \tfrac{1}{2}x - \tfrac{1}{4} \sin 2x + C$

REPEATED INTEGRATION BY PARTS

It is sometimes necessary to use integration by parts more than once in the same problem.

▶ **Example 4** Evaluate $\int x^2 e^{-x}\, dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\, dx, \quad du = 2x\, dx, \quad v = \int e^{-x}\, dx = -e^{-x}$$

so that from (3)

$$\int x^2 e^{-x}\,dx = \int u\,dv = uv - \int v\,du$$
$$= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx$$
$$= -x^2e^{-x} + 2\int xe^{-x}\,dx \qquad (4)$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C$$
$$= -(x^2 + 2x + 2)e^{-x} + C \blacktriangleleft$$

▸ **Example 5** Evaluate $\int e^x \cos x\, dx$.

Solution. Let

$$u = \cos x, \quad dv = e^x\, dx, \quad du = -\sin x\, dx, \quad v = \int e^x\, dx = e^x$$

Thus,

$$\int e^x \cos x\, dx = \int u\, dv = uv - \int v\, du = e^x \cos x + \int e^x \sin x\, dx \qquad (5)$$

Since the integral $\int e^x \sin x\, dx$ is similar in form to the original integral $\int e^x \cos x\, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x\, dx, \quad du = \cos x\, dx, \quad v = \int e^x\, dx = e^x$$

Thus,

$$\int e^x \sin x\, dx = \int u\, dv = uv - \int v\, du = e^x \sin x - \int e^x \cos x\, dx$$

Together with Equation (5) this yields

$$\int e^x \cos x\, dx = e^x \cos x + e^x \sin x - \int e^x \cos x\, dx \qquad (6)$$

which is an equation we can solve for the unknown integral. We obtain

$$2\int e^x \cos x\, dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x\, dx = \tfrac{1}{2}e^x \cos x + \tfrac{1}{2}e^x \sin x + C \blacktriangleleft$$

8. Find $\int x^2 \sin x\, dx$.

Let $u = x^2$, $dv = \sin x\, dx$. Then $du = 2x\, dx$, $v = -\cos x$, and

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$$

For the resulting integral, let $u = x$ and $dv = \cos x\, dx$. Then $du = dx$, $v = \sin x$, and

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2\left(x \sin x - \int \sin x\, dx\right) = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

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9. Find $\int x^3 e^{2x}\, dx$.

Let $u = x^3$, $dv = e^{2x}\, dx$. Then $du = 3x^2\, dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\, dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\int x^2 e^{2x}\, dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\, dx$. Then $du = 2x\, dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\, dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\left(\tfrac{1}{2}x^2 e^{2x} - \int x e^{2x}\, dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\int x e^{2x}\, dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\, dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\, dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\left(\tfrac{1}{2}x e^{2x} - \tfrac{1}{2}\int e^{2x}\, dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{4}x e^{2x} - \tfrac{3}{8}e^{2x} + C$$

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Find $\int x \sin x \, dx$.

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = \int dv$

$= \int \sin x \, dx$

$v = -\cos x$

LIATE

$uv - \int v \, du$

$= x(-\cos x) - \int -\cos x \, dx$

$= -x \cos x - (-\sin x) + C$

$= -x \cos x + \sin x + C$

3)

LIATE

Evaluate $\int xe^x\,dx$.

$u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = \int e^x dx = e^x$

$uv - \int v\,du$

$= xe^x - \int e^x dx$

$= xe^x - e^x + c$ //

$e^x(x-1) + c$

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11) $\int x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{dx}{x} \qquad v = \frac{x^4}{4}$

$uv - \int v \, du$

$= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{dx}{x}$

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx$

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \frac{x^4}{4} + c$

$= \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C$ or $= \frac{1}{4} x^4 \left(\ln x - \frac{1}{4} \right) + c$ or $\frac{1}{16} x^4 (4 \ln x - 1) + c$

12) $\int (7-x)\, e^{x/2}\, dx$

$u = 7-x \qquad dv = e^{\frac{1}{2}x}\, dx$

$du = -dx \qquad v = 2e^{\frac{1}{2}x}$

$uv - \int v\, du$

$= (7-x)\, 2e^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x} \cdot -dx$

$= (14-2x)e^{\frac{1}{2}x} + 2\int e^{\frac{1}{2}x}\, dx$

$= (14-2x)e^{\frac{1}{2}x} + 4e^{\frac{1}{2}x} + c$

$= e^{\frac{1}{2}x}\left((14-2x)+4\right] + c$ or $\boxed{e^{\frac{1}{2}x}(18-2x) + c}$ or $\boxed{18e^{\frac{1}{2}x} - 2xe^{\frac{1}{2}x} + c}$

	D	I
+	$7-x$	$e^{\frac{1}{2}x}$
−	-1	$2e^{\frac{1}{2}x}$
+	0	$4e^{\frac{1}{2}x}$

$$\int (2x+1)\sin 4x\,dx$$

$u = 2x+1 \qquad dv = \sin 4x\,dx$

$du = 2\,dx \qquad v = -\frac{1}{4}\cos 4x$

$uv - v\,du$

$$= (2x+1)\left(-\frac{1}{4}\cos 4x\right) - \int -\frac{1}{4}\cos 4x(2\,dx)$$

$$= -\frac{1}{4}(2x+1)\cos 4x + \frac{1}{2}\int \cos 4x\,dx$$

$$= -\frac{1}{4}(2x+1)\cos 4x + \frac{1}{2}\left(\frac{1}{4}\sin 4x\right) + C$$

$$= -\frac{1}{4}(2x+1)\cos 4x + \frac{1}{8}\sin 4x + C$$

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$$\int x\cos 4x\,dx$$

$u = x \qquad dv = \cos 4x\,dx$

$du = dx \qquad v = \frac{1}{4}\sin 4x$

$$uv - \int v\,du$$

$$= (x)\left(\frac{1}{4}\sin 4x\right) - \int \frac{1}{4}\sin 4x\,dx$$

$$= \frac{1}{4}x\sin 4x - \frac{1}{4}\int \sin 4x\,dx$$

$$= \frac{1}{4}x\sin 4x - \frac{1}{4}\left(-\frac{1}{4}\cos 4x\right) + c$$

$$= \frac{1}{4}x\sin(4x) + \frac{1}{16}\cos 4x + c \quad \text{or} \quad \frac{1}{16}\left(4x\sin(4x) + \cos(4x)\right) + c$$

4) Evaluate $\int \ln x \, dx$.

$u = \ln x$

$du = \frac{dx}{x}$ or $\frac{1}{x} dx$

$dv = dx$

$v = \int dx = x$

$uv - \int v \, du$

$= \ln x (x) - \int x \frac{1}{x} dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

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Find $\int x^3 e^{x^2}\,dx$.

$\int x^2 \left(e^{x^2} x\,dx\right)$

$u = x^2$

$du = 2x\,dx$

$dv = e^{x^2} x\,dx$

$v = \int dv = \frac{1}{2}\int e^{x^2} x\,dx$

$V = \frac{1}{2} e^{x^2}$

$e^{x^2} = e^{x^2} (x\,dx)$

$e^u =$

$u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$uv - \int v\,du$

$= x^2 \frac{1}{2} e^{x^2} - \int \frac{1}{2} e^{x^2}\, 2x\,dx$

$= \frac{1}{2} x^2 e^{x^2} - \int e^{x^2} x\,dx$

$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$

Find $\int x^2 \ln x \, dx$.

$u = \ln x \quad dv = x^2 dx$

$du = \frac{1}{x} dx \quad v = \int x^2 dx$

$v = \frac{x^3}{3}$

$uv - \int v \, du$

$= \ln x \frac{x^3}{3} - \int \frac{x^3}{3} \frac{dx}{x}$

$= \frac{x^3}{3} \ln x - \int \frac{x^2 dx}{3}$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \frac{x^3}{3} + c$

$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + c$

Ap

Find $\int x\sqrt{1+x}\,dx$.

$(1+x)$

$u = 1+x$

$du = dx$

$= 1\,dx = dx$

$u = x \qquad dv = \sqrt{1+x}\,dx$

$du = dx \qquad = \int (1+x)^{1/2}\,dx$

$= \int u^{1/2}\,du$

$= \frac{u^{3/2}}{3/2}$ or $\frac{2}{3}u^{3/2}$

$v = \frac{2}{3}(1+x)^{3/2}$

$\frac{u^{3/2+1}}{3/2+1} = \frac{u^{5/2}}{5/2}$

$= \frac{2}{5}u^{5/2}$

$uv - \int v\,du$

$= x\left(\frac{2}{3}(1+x)^{3/2}\right) - \int \frac{2}{3}(1+x)^{3/2}\,dx$

$= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2}\,dx$

$= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\left(\frac{2}{5}(1+x)^{5/2}\right) + C$

$= \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C$

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Find $\int \arcsin x \, dx$.

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$uv - \int v \, du$

$= \arcsin x (x) - \int x \frac{1}{\sqrt{1-x^2}} dx$

$= x \arcsin x - -(1-x^2)^{1/2}$

$= x \arcsin x + (1-x^2)^{1/2} + C$

or

$x \arcsin x + \sqrt{1-x^2} + C$

$\int \frac{x \, dx}{\sqrt{1-x^2}}$

$= \int x (1-x^2)^{-1/2} dx$

$u = 1 - x^2$

$du = -2x \, dx$

$dx = \frac{du}{-2x}$

$\int x u^{-1/2} \frac{du}{-2x}$

$= -\frac{1}{2} \int u^{-\frac{1}{2}} du$

$= -\frac{1}{2} \frac{u^{1/2}}{1/2} = -u^{1/2}$

$= -(1-x^2)^{1/2}$

Evaluate $\int x^2 e^{-x}\, dx$.

LIATE

$u = x^2 \quad dv = e^{-x} dx$

$du = 2x\, dx \quad v = \int e^{-x} dx$

$v = -e^{-x}$

$uv - \int v\, du$

$= x^2 e^{-x} - \int -e^{-x}\, 2x\, dx$

$= -x^2 e^{-x} - 2 \int -x e^{-x} dx$

$= -x^2 e^{-x} + 2(-x e^{-x} - e^{-x}) + C$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

$= -e^{-x}(x^2 + 2x + 2) + C$

$\int x e^{-x} dx$

integration by parts

$u = x \quad dv = e^{-x} dx$

$du = dx \quad v = -e^{-x}$

$uv - \int v\, du$

$= -x e^{-x} - \int -e^{-x} dx$

$= -x e^{-x} - e^{-x}$

SUMMARY: COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax}\,dx, \quad \int x^n \sin ax\,dx, \quad \text{or} \quad \int x^n \cos ax\,dx$$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

2. For integrals of the form

$$\int x^n \ln x\,dx, \quad \int x^n \arcsin ax\,dx, \quad \text{or} \quad \int x^n \arctan ax\,dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx\,dx \quad \text{or} \quad \int e^{ax} \cos bx\,dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

A TABULAR METHOD FOR REPEATED INTEGRATION BY PARTS

Integrals of the form

$$\int p(x)f(x)\,dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts in which u is taken to be $p(x)$ or one of its derivatives at each stage. Since du is computed by differentiating u, the repeated differentiation of $p(x)$ will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called *tabular integration by parts*.

INTEGRAL CALCULUS

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

INTEGRAL CALCULUS

This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x) \cos x \, dx$.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\int (x^2 - x) \cos x \, dx = (x^2 - x) \sin x + (2x - 1) \cos x - 2 \sin x + C$$
$$= (x^2 - x - 2) \sin x + (2x - 1) \cos x + C$$

INTEGRAL CALCULUS

$\int (x^2 - x)\cos x\, dx.$

Using Tabular or D I

	Derivative	Integral
+	$x^2 - x$	$\cos x$
−	$2x - 1$	$\sin x$
+	2	$-\cos x$
−	0	$-\sin x$

$$= (x^2 - x)\sin x + (2x-1)\cos x - 2\sin x + C$$

$$= x^2 \sin x - x\sin x - 2\sin x + (2x-1)\cos x + C$$

$$= (x^2 - x - 2)\sin x + (2x-1)\cos x + C$$

Example 6 $\int x^2\sqrt{x-1}\,dx$

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

$$\int x^2\sqrt{x-1}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C \blacktriangleleft$$

Example 6 $\int x^2\sqrt{x-1}\,dx$

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C \blacktriangleleft$$

INTEGRAL CALCULUS

Ex 3

$$\int (3x^2 - x + 2)e^{-x}\,dx. \text{ Then}$$

diff.		antidiff.
$3x^2 - x + 2$		e^{-x}
	$\searrow +$	
$6x - 1$		$-e^{-x}$
	$\searrow -$	
6		e^{-x}
	$\searrow +$	
0		$-e^{-x}$

$$I = \int (3x^2 - x + 2)e^{-x} = -(3x^2 - x + 2)e^{-x} - (6x - 1)e^{-x} - 6e^{-x} + C = -e^{-x}[3x^2 + 5x + 7] + C.$$

Let I denote $\int 4x^4 \sin 2x \, dx$. Then

diff.		antidiff.
$4x^4$		$\sin 2x$
	$\searrow +$	
$16x^3$		$-\frac{1}{2}\cos 2x$
	$\searrow -$	
$48x^2$		$-\frac{1}{4}\sin 2x$
	$\searrow +$	
$96x$		$\frac{1}{8}\cos 2x$
	$\searrow -$	
96		$\frac{1}{16}\sin 2x$
	$\searrow +$	
0		$-\frac{1}{32}\cos 2x$

$$I = \int 4x^4 \sin 2x \, dx = (-2x^4 + 6x^2 - 3)\cos 2x + (4x^3 - 6x)\sin 2x + C.$$

51. Let I denote $\int e^{ax}\sin bx\,dx$. Then

diff.		antidiff.
e^{ax}		$\sin bx$
	$\searrow +$	
ae^{ax}		$-\frac{1}{b}\cos bx$
	$\searrow -$	
a^2e^{ax}		$-\frac{1}{b^2}\sin bx$

$$I = \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I, \text{ so } I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

INTEGRAL CALCULUS

Integration by Parts (some cases)

1) $\int e^x \cos x \, dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x \, dx \qquad v = e^x$

$uv - \int v \, du$

$e^x \cos x - \int e^x(-\sin x) \, dx$

$e^x \cos x + \int e^x \sin x \, dx$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x \, dx \qquad v = e^x$

$e^x \cos x + e^x \sin x - \int e^x \cos \, dx$

Same as origina

$\underbrace{\int e^x \cos x \, dx}_{\text{same}} = e^x \cos x + e^x \sin x - \underbrace{\int e^x \cos x \, dx}_{\text{same}}$

$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$

$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C$

$$\int e^x \cos x \, dx = \frac{e^x \cos x}{2} + \frac{e^x \sin x}{2} + C$$

INTEGRAL CALCULUS

Using DI method case 2

$$\int e^x \cos x \, dx$$

	D	I
+	$\cos x$	e^x
−	$-\sin x$	e^x
+	$-\cos x$	e^x

$$e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C$$

$$\int e^x \cos x \, dx = \boxed{\frac{e^x \cos x}{2} + \frac{e^x \sin x}{2} + C}$$

INTEGRAL CALCULUS

Ex 2 $\int e^{2x}\sin 3x\,dx$

	D	I
+	$\sin 3x$	e^{2x}
-	$3\cos 3x$	$\frac{1}{2}e^{2x}$
+	$-9\sin 3x$	$\frac{1}{4}e^{2x}$

Case 2 similar to given

$$\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \int \frac{9}{4}e^{2x}\sin 3x\,dx$$

$$\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\,dx$$

$$\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\,dx$$

$$\int e^{2x}\sin 3x\,dx + \frac{9}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x + C$$

$$\frac{13}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x + C$$

$$\int e^{2x}\sin 3x\,dx = \frac{4}{26}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C$$

$$= \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C$$

INTEGRAL CALCULUS

DI Case 3

3) $\int \frac{\ln x}{\sqrt{x}}\,dx$

	D	I
+	$\ln x$	$\frac{1}{\sqrt{x}}$ or $\frac{1}{x^{\frac{1}{2}}} \to x^{-\frac{1}{2}}$
−	$\frac{1}{x}$	$2x^{\frac{1}{2}}$

case 3 if integrable

$2x^{\frac{1}{2}}\ln x - \int \frac{1}{x} 2x^{\frac{1}{2}}$

$2x^{\frac{1}{2}}\ln x - 2\int \frac{x^{\frac{1}{2}}}{x}\,dx$

$2x^{\frac{1}{2}}\ln x - 2\int x^{-\frac{1}{2}}\,dx$

$2x^{\frac{1}{2}}\ln x - 2\left[2x^{\frac{1}{2}}\right] + c$

$2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}} + c$

$2\sqrt{x}\ln x - 4\sqrt{x} + c$ or

$2\sqrt{x}(\ln x - 2) + c$

INTEGRAL CALCULUS

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4) $\int x^2 \ln x \, dx$

	D	I
+	$\ln x$	x^2
−	$\frac{1}{x}$	$\frac{x^3}{3}$

case 3 integrable

$$= \frac{x^3}{3}\ln x - \int \frac{1}{x}\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + C$$

$$= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$$

or

$$\boxed{\frac{1}{9}x^3(3\ln x - 1) + C}$$

INTEGRAL CALCULUS

May

5) $\int \sin\sqrt{x}\,dx$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$\int \sin u \; 2u\,du$

$2\int u \sin u\,du$

	D	I
+	u	$\sin u$
−	1	$-\cos u$
+	0	$-\sin u$

$2(-u\cos u + \sin u) + C$

$-2u\cos u + 2\sin u + C$

$-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

6) $\int \ln(2+x^2)\,dx$

	D	I
+	$\ln(2+x^2)$	1
−	$\frac{2x}{2+x^2}$	x

Integrate

$x\ln(2+x^2) - \int \frac{2x^2}{2+x^2}\,dx$

$$\begin{array}{r} 2 \\ 2+x^2 \overline{)\,2x^2\,} \\ -\,2x^2+4 \\ \hline -4 \end{array}$$

$2 - \frac{4}{2+x^2}\,dx$

$-\int\left(2 - \frac{4}{2+x^2}\right)dx$

$-\int 2\,dx + \int \frac{4\,dx}{2+x^2}$

$-\int 2\,dx + 4\int \frac{dx}{2+x^2}$

$a = \sqrt{2}$

$u = x$

$-2x + 4\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right)$

$-2x + \frac{4}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}$

$\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

$$\boxed{x\ln(2+x^2) - 2x + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}$$

REDUCTION FORMULAS

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{9}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{10}$$

proof

To illustrate how such formulas can be obtained, let us derive (10). We begin by writing $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$ and letting

$$u = \cos^{n-1} x \qquad dv = \cos x\, dx$$

$$du = (n-1)\cos^{n-2} x(-\sin x)\, dx \qquad v = \sin x$$

$$= -(n-1)\cos^{n-2} x \sin x\, dx$$

so that

$$\int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx = \int u\, dv = uv - \int v\, du$$

$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx$$

Moving the last term on the right to the left side yields

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx$$

Example Evaluate $\int \cos^4 x\,dx$.

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \qquad (10)$$

Solution. From (10) with $n = 4$

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx$$

Now apply (10) with $n = 2$.

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx\right)$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$$

1) $\int \cos^4 x \, dx$ $\qquad \int \cos^n x \, dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$

$n=4$

$$= \frac{1}{4}\cos^{4-1}x \sin x + \frac{4-1}{4}\int \cos^{4-2}x \, dx$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x \, dx$$

$n=2$ $\quad \frac{1}{2}\cos^{2-1}x \sin x + \frac{2-1}{2}\int \cos^{2-2} dx$

$$\frac{3}{4}\left[\frac{1}{2}\cos^1 x \sin x + \frac{1}{2}\int dx\right]$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right) + C$$

$$\boxed{= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C}$$

INTEGRAL CALCULUS

(a) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(b) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(c) $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$

65. (a) $\int \tan^4 x \, dx$ (b) $\int \sec^4 x \, dx$ (c) $\int x^3 e^x \, dx$

TRANSFORMATIONS of TRIGONOMETRIC FUNCTIONS

The transformation of the trigonometric functions are divided into two major parts, they are the following:

Part 1: Powers of Sine and Cosine

Part 2: Powers of Tangent and Secant and
Powers of Cotangent and Cosecant

Part 1: Powers of Sine and Cosine

$Case\,1$: Consider the integrand

$$\int \sin^{n} u\, du \text{ or } \int \cos^{n} u\, du \quad \text{where n is any positive odd integer}$$

$$\text{use the identity}: \sin^{n} u + \cos^{n} u = 1$$

$Case\,2$: Consider the integrand

$$\int \sin^{n} u\, du \text{ or } \int \cos^{n} u\, du \quad \text{where n is any positive even integer}$$

$$\text{use the identity}: \sin^{n} u = \frac{1}{2}\left(1 - \cos 2u\right)$$

$$\cos^{n} u = \frac{1}{2}\left(1 + \cos 2u\right)$$

INTEGRAL CALCULUS

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \qquad (1)$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \qquad (2)$$

In the case where $n = 2$, these formulas yield

$$\int \sin^2 x\,dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x - \tfrac{1}{2}\sin x \cos x + C \qquad (3)$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x \cos x + C \qquad (4)$$

INTEGRAL CALCULUS

Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x) \tag{5–6}$$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2\sin^2 x \quad \text{and} \quad \cos 2x = 2\cos^2 x - 1$$

These identities yield

$$\int \sin^2 x\, dx = \tfrac{1}{2}\int (1 - \cos 2x)\, dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C \tag{7}$$

$$\int \cos^2 x\, dx = \tfrac{1}{2}\int (1 + \cos 2x)\, dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C \tag{8}$$

Observe that the antiderivatives in Formulas (3) and (4) involve both sines and cosines, whereas those in (7) and (8) involve sines alone. However, the apparent discrepancy is easy to resolve by using the identity

$$\sin 2x = 2 \sin x \cos x$$

to rewrite (7) and (8) in forms (3) and (4), or conversely.

In the case where $n = 3$, the reduction formulas for integrating $\sin^3 x$ and $\cos^3 x$ yield

$$\int \sin^3 x\,dx = -\tfrac{1}{3}\sin^2 x\cos x + \tfrac{2}{3}\int \sin x\,dx = -\tfrac{1}{3}\sin^2 x\cos x - \tfrac{2}{3}\cos x + C \qquad (9)$$

$$\int \cos^3 x\,dx = \tfrac{1}{3}\cos^2 x\sin x + \tfrac{2}{3}\int \cos x\,dx = \tfrac{1}{3}\cos^2 x\sin x + \tfrac{2}{3}\sin x + C \qquad (10)$$

$$\int \sin^3 x\, dx = \tfrac{1}{3}\cos^3 x - \cos x + C \tag{11}$$

$$\int \cos^3 x\, dx = \sin x - \tfrac{1}{3}\sin^3 x + C \tag{12}$$

$$\int \sin^4 x\, dx = \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \tag{13}$$

$$\int \cos^4 x\, dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \tag{14}$$

INTEGRATING PRODUCTS OF SINES AND COSINES

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by one of the three procedures stated in Table 7.3.1, depending on whether m and n are odd or even.

▶ Example 2 Evaluate

$$\text{(a)} \int \sin^4 x \cos^5 x \, dx \qquad \text{(b)} \int \sin^4 x \cos^4 x \, dx$$

Table 7.3.1

INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Solution (a). Since $n = 5$ is odd, we will follow the first procedure in Table 7.3.1:

$$\begin{aligned}
\int \sin^4 x \cos^5 x\, dx &= \int \sin^4 x \cos^4 x \cos x\, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x\, dx \\
&= \int u^4 (1 - u^2)^2\, du \\
&= \int (u^4 - 2u^6 + u^8)\, du \\
&= \tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9 + C \\
&= \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C
\end{aligned}$$

INTEGRAL CALCULUS

Solution (b). Since $m = n = 4$, both exponents are even, so we will follow the third procedure in Table 7.3.1:

$$\int \sin^4 x \cos^4 x\, dx = \int (\sin^2 x)^2 (\cos^2 x)^2\, dx$$

$$= \int \left(\tfrac{1}{2}[1 - \cos 2x]\right)^2 \left(\tfrac{1}{2}[1 + \cos 2x]\right)^2\, dx$$

$$= \tfrac{1}{16} \int (1 - \cos^2 2x)^2\, dx$$

$$= \tfrac{1}{16} \int \sin^4 2x\, dx$$

Note that this can be obtained more directly from the original integral using the identity $\sin x \cos x = \frac{1}{2} \sin 2x$.

$$= \tfrac{1}{32} \int \sin^4 u\, du$$

$u = 2x$
$du = 2\,dx$ or $dx = \frac{1}{2}\,du$

$$= \tfrac{1}{32} \left(\tfrac{3}{8} u - \tfrac{1}{4} \sin 2u + \tfrac{1}{32} \sin 4u\right) + C$$

Formula (13)

$$= \tfrac{3}{128} x - \tfrac{1}{128} \sin 4x + \tfrac{1}{1024} \sin 8x + C$$ ◄

INTEGRAL CALCULUS

Integrals of the form

$$\int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx \tag{15}$$

can be found by using the trigonometric identities

$$\sin\alpha \cos\beta = \tfrac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)] \tag{16}$$

$$\sin\alpha \sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \tag{17}$$

$$\cos\alpha \cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \tag{18}$$

to express the integrand as a sum or difference of sines and cosines.

▶ **Example 3** Evaluate $\int \sin 7x \cos 3x \, dx$.

Solution. Using (16) yields

$$\int \sin 7x \cos 3x \, dx = \frac{1}{2} \int (\sin 4x + \sin 10x) \, dx = -\frac{1}{8} \cos 4x - \frac{1}{20} \cos 10x + C$$ ◀

INTEGRAL CALCULUS

1–52 Evaluate the integral.

1. $\int \cos^3 x \sin x \, dx$

3. $\int \sin^2 5\theta \, d\theta$

INTEGRAL CALCULUS

7. $\int \sin ax \cos ax \, dx$

9. $\int \sin^2 t \cos^3 t \, dt$

INTEGRAL CALCULUS

13. $\int \sin 2x \cos 3x \, dx$

15. $\int \sin x \cos(x/2) \, dx$

INTEGRAL CALCULUS

Case 3: If the intergrand contains the product of sine and cosine functions of the form $\int \sin^{n} u \cdot \cos^{m} u \, du$ where at least one of the m or n is a positive odd integer, employ the same technique as that of Case 1.

Case 4: If the intergrand contains the product of sine and cosine functions of the form $\int \sin^{n} u \cdot \cos^{m} u \, du$ where both m or n is a positive even integer, employ the same technique as that of Case 2.

Case 5 : If the intergrand has any of the following form :

$$\int \sin au \cdot \cos bu \, du; \quad \int \cos au \cdot \cos bu \, du; \quad \int \sin au \cdot \sin bu \, du$$

Use the following transformations :

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left[-\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$where \ \alpha = \text{au and } \beta = \text{bu}$$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x \, dx$$

can be evaluated by one of the three procedures stated in Table 7.3.2, depending on whether m and n are odd or even.

Table 7.3.2

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

▸ Example 4 Evaluate

(a) $\int \tan^2 x \sec^4 x \, dx$ (b) $\int \tan^3 x \sec^3 x \, dx$ (c) $\int \tan^2 x \sec x \, dx$