# CALCULUS 2

## INTEGRATION TECHNIQUES

### Integration by parts

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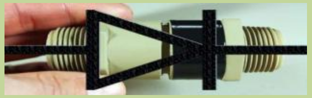
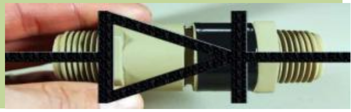


# OBJECTIVES



- to evaluate integrals using integration by parts
- integrate functions using repeated integration by parts
- Integrate functions using tabular form

# INTEGRAL CALCULUS

**Integration by Parts**:

The most useful among the techniques of integration is the integration by parts.

It is derived from the differentials of the product of two factors. If u and v are both differentiable functions of x, then

$$d(uv) = udv + vdu$$

# INTEGRAL CALCULUS

$$d(uv) = udv + vdu$$

By transposition,

$$udv = d(uv) - vdu$$

Integrating both sides of the equation, we have

$$\int udv = uv - \int vdu$$

# INTEGRAL CALCULUS

The integral $\int \mathrm{udv}$ is expressed in terms of another

integral $\int \mathrm{vdu}$ which must be simpler than the given integral,

and is easier to evaluate.

Thus, given an integrand, a factor may be set as u, which is differentiable, and the other part as dv where its integral must exist. The process can be used repeatedly.

# INTEGRAL CALCULUS

The technique is used in integrating odd powers of :

- odd powers secant, cosecant, hyperbolic secant and hyperbolic cosecant like ,

$$\int \sec^3 4x\,dx \qquad \int x\,\mathrm{csch}^5 x^2\,dx$$

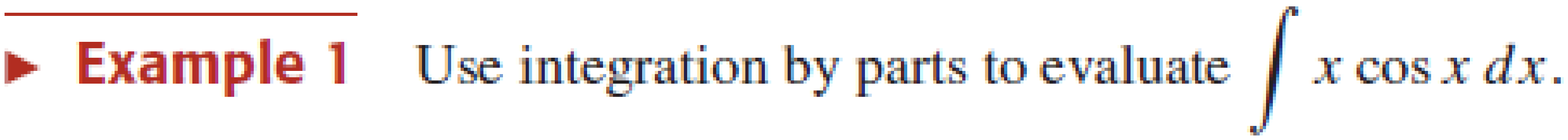
- inverses of trigonometric and hyperbolic functions like,

$$\int \sin^{-1} 2x\,dx \qquad \int x \cosh^{-1} 3x\,dx$$

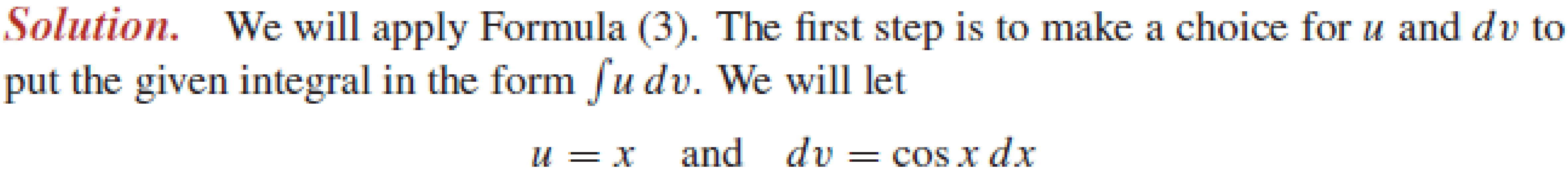
- products of transcendental /algebraic functions like

$$\int x^2 \sin 4x\,dx \qquad \int e^{2x} \cos x\,dx$$

# INTEGRAL CALCULUS



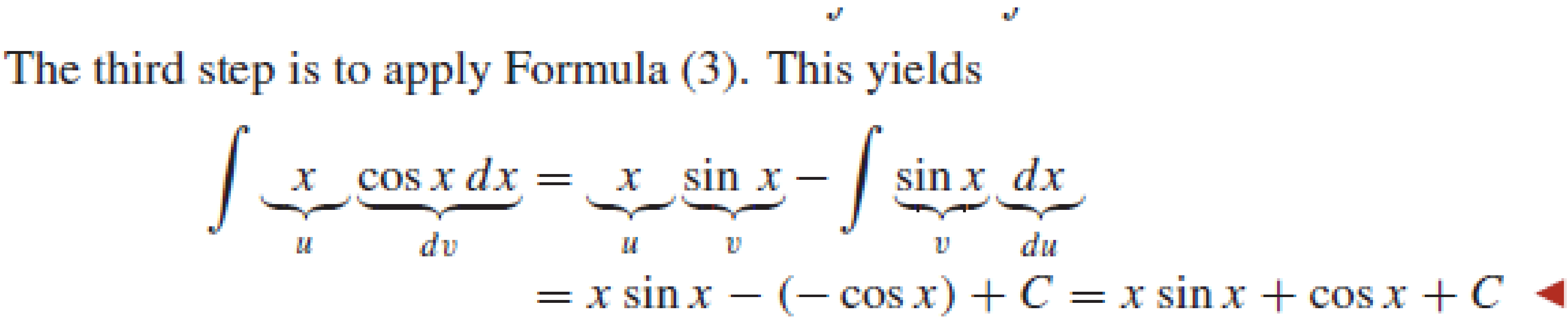
**Example 1** Use integration by parts to evaluate $\int x \cos x \, dx$.



*Solution.* We will apply Formula (3). The first step is to make a choice for $u$ and $dv$ to put the given integral in the form $\int u \, dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x \, dx$$

The second step is to compute *du* from *u* and *v* from *dv*. This yields

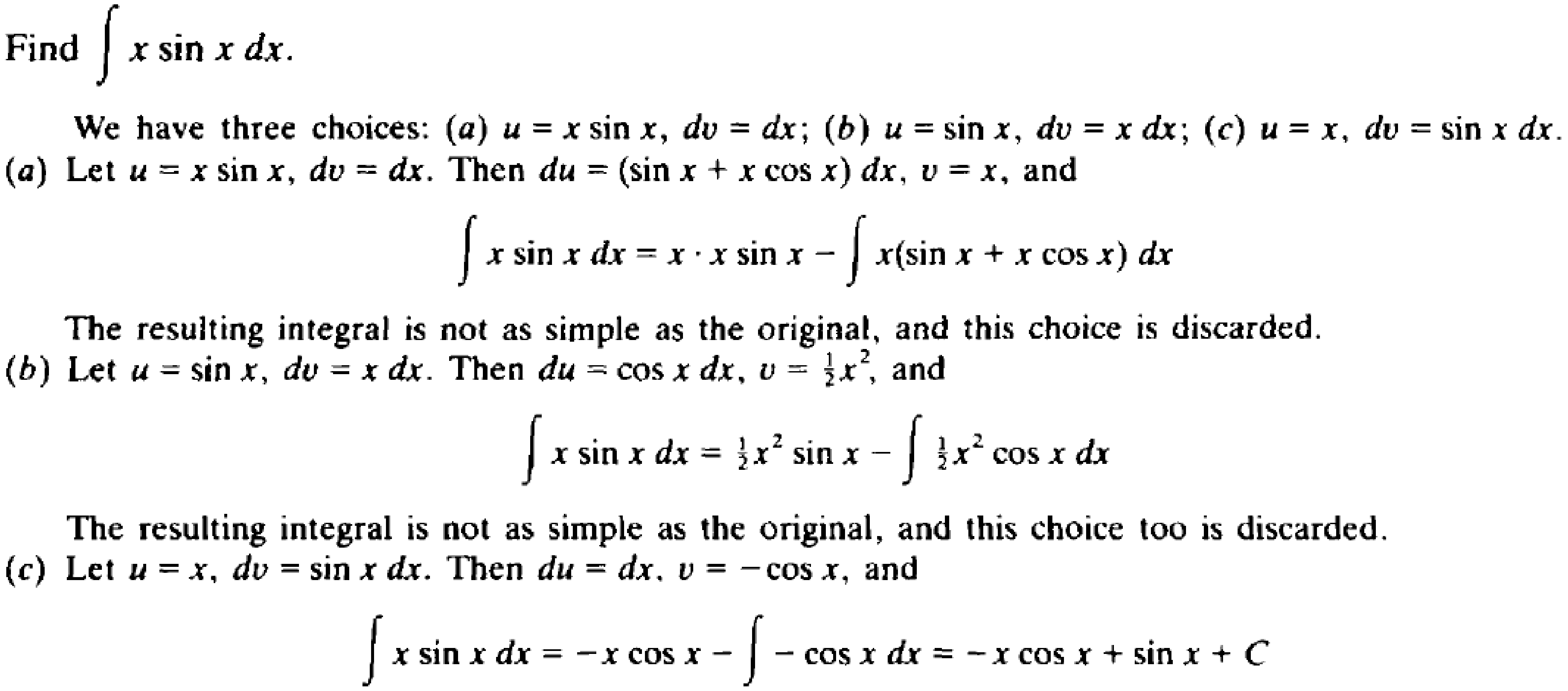


The third step is to apply Formula (3). This yields

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = \underbrace{x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \underbrace{dx}_{du}$$

$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C \blacktriangleleft$$

# INTEGRAL CALCULUS



Find $\int x \sin x \, dx$.

We have three choices: (*a*) $u = x \sin x$, $dv = dx$; (*b*) $u = \sin x$, $dv = x\,dx$; (*c*) $u = x$, $dv = \sin x \, dx$.

(*a*) Let $u = x \sin x$, $dv = dx$. Then $du = (\sin x + x \cos x)\,dx$, $v = x$, and

$$\int x \sin x \, dx = x \cdot x \sin x - \int x(\sin x + x \cos x)\,dx$$

The resulting integral is not as simple as the original, and this choice is discarded.

(*b*) Let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x \, dx$, $v = \frac{1}{2}x^2$, and

$$\int x \sin x \, dx = \tfrac{1}{2}x^2 \sin x - \int \tfrac{1}{2}x^2 \cos x \, dx$$

The resulting integral is not as simple as the original, and this choice too is discarded.

(*c*) Let $u = x$, $dv = \sin x \, dx$. Then $du = dx$, $v = -\cos x$, and

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \sin x + C$$

The integral of $\ln(x)$ can be calculated using integration by parts. Let's denote $u = \ln(x)$ and $dv = dx$. Then, we have $du = \frac{1}{x}dx$ and $v = x$.

Using the integration by parts formula:

$$\int \ln(x)\,dx = uv - \int v\,du$$

$$= x\ln(x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C$$

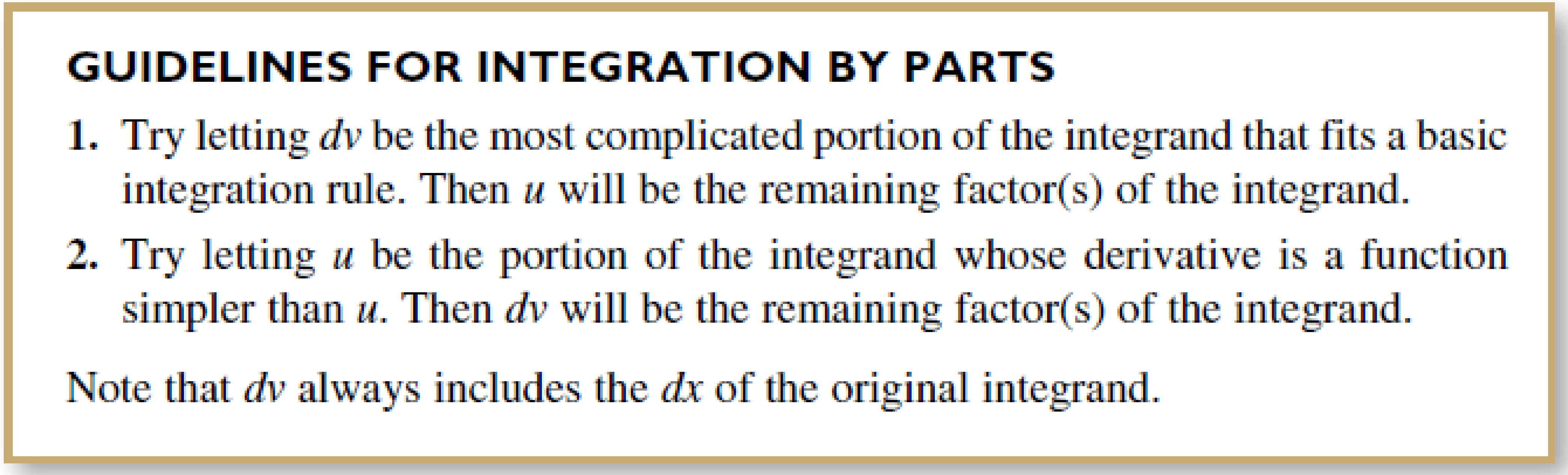
# INTEGRAL CALCULUS

## GUIDELINES FOR INTEGRATION BY PARTS

- The main goal in integration by parts is to choose $u$ and $dv$ to obtain a new integral that is easier to evaluate than the original.
- In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice.
- A strategy that often works is to choose $u$ and $dv$ so that $u$ becomes “simpler” when differentiated, while leaving a $dv$ that can be readily integrated to obtain $v$.

There is another useful strategy for choosing $u$ and $dv$ that can be applied when the integrand is a product of two functions from *different* categories in the list

**L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential



## GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting $dv$ be the most complicated portion of the integrand that fits a basic integration rule. Then $u$ will be the remaining factor(s) of the integrand.
2. Try letting $u$ be the portion of the integrand whose derivative is a function simpler than $u$. Then $dv$ will be the remaining factor(s) of the integrand.

Note that $dv$ always includes the $dx$ of the original integrand.

## **EXAMPLE 1** Integration by Parts

Find $\int xe^x\,dx$.

**Solution** To apply integration by parts, you need to write the integral in the form $\int u\,dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x\,dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\underbrace{(xe^x\,dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines on the preceding page suggest the first option because the derivative of $u = x$ is simpler than $x$, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

$$dv = e^x\,dx \quad \Rightarrow \quad v = \int dv = \int e^x\,dx = e^x$$

$$u = x \quad \Rightarrow \quad du = dx$$

Now, integration by parts produces

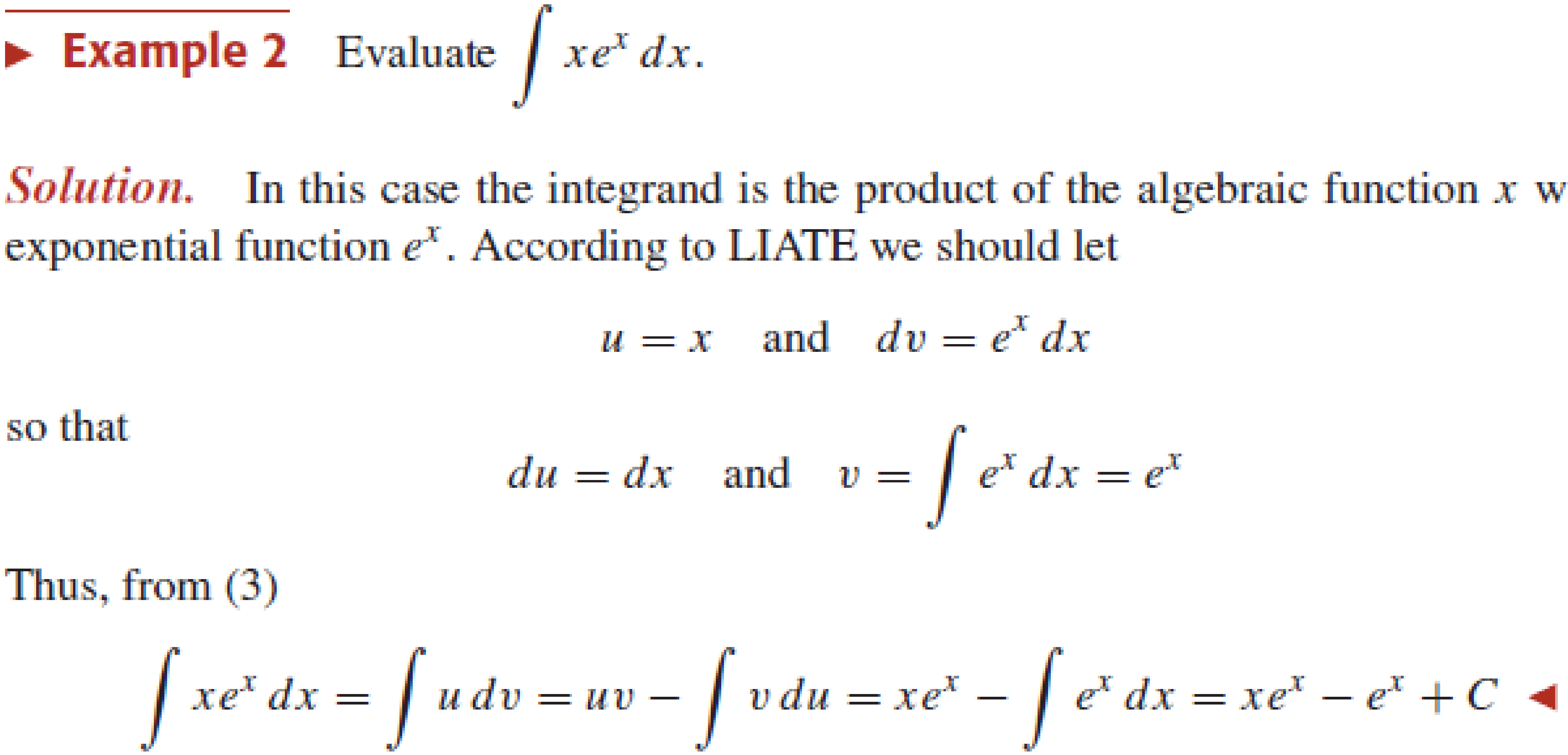
$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

# INTEGRAL CALCULUS



**Example 2** Evaluate $\int xe^x\,dx$.

*Solution.* In this case the integrand is the product of the algebraic function $x$ with the exponential function $e^x$. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

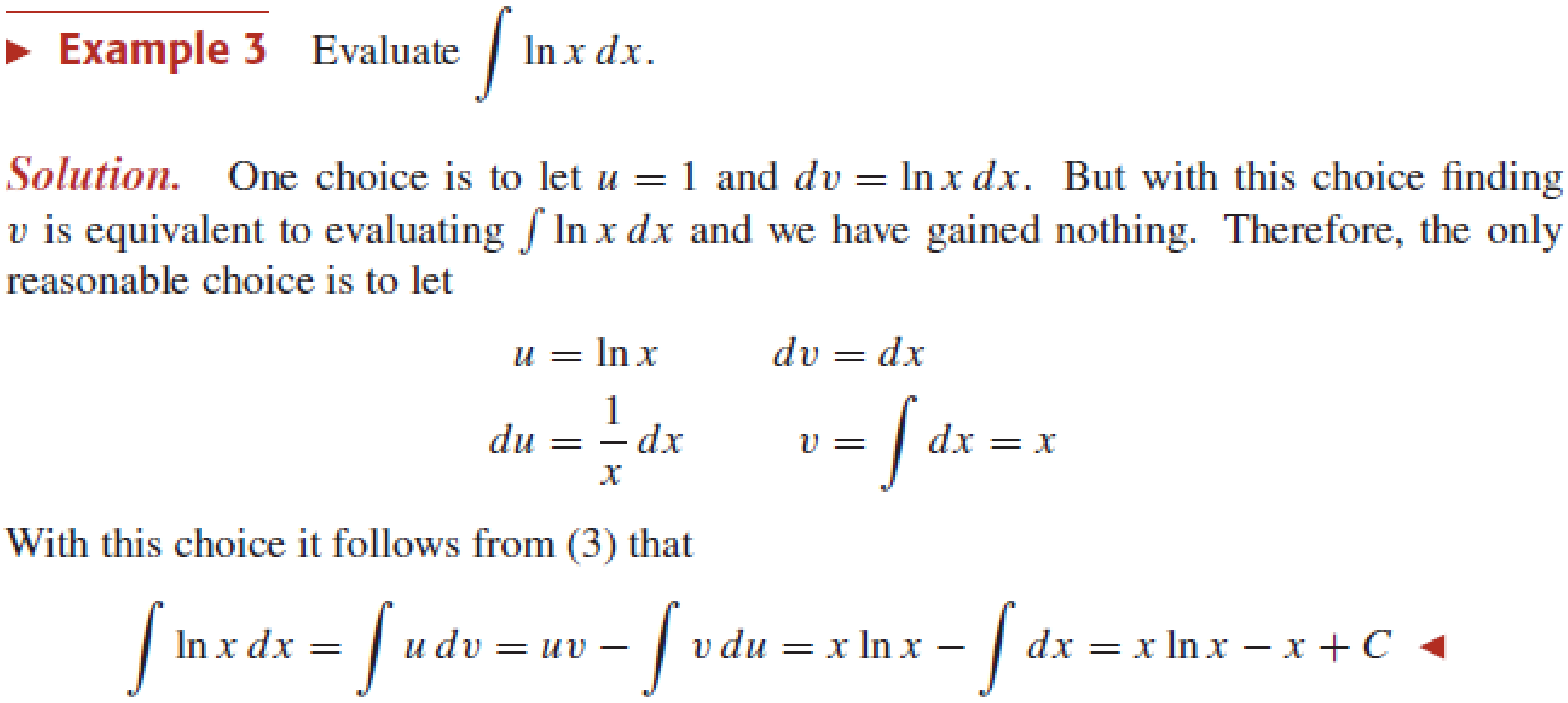
so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C \blacktriangleleft$$

# INTEGRAL CALCULUS



▸ **Example 3** Evaluate $\int \ln x \, dx$.

*Solution.* One choice is to let $u = 1$ and $dv = \ln x \, dx$. But with this choice finding $v$ is equivalent to evaluating $\int \ln x \, dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x} dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x \, dx = \int u \, dv = uv - \int v \, du = x \ln x - \int dx = x \ln x - x + C \blacktriangleleft$$

## **EXAMPLE 2** Integration by Parts

Find $\int x^2 \ln x \, dx$.

**Solution** In this case, $x^2$ is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should let $dv = x^2\,dx$.

$$dv = x^2\,dx \quad \Rightarrow \quad v = \int x^2\,dx = \frac{x^3}{3}$$

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$$

Integration by parts produces

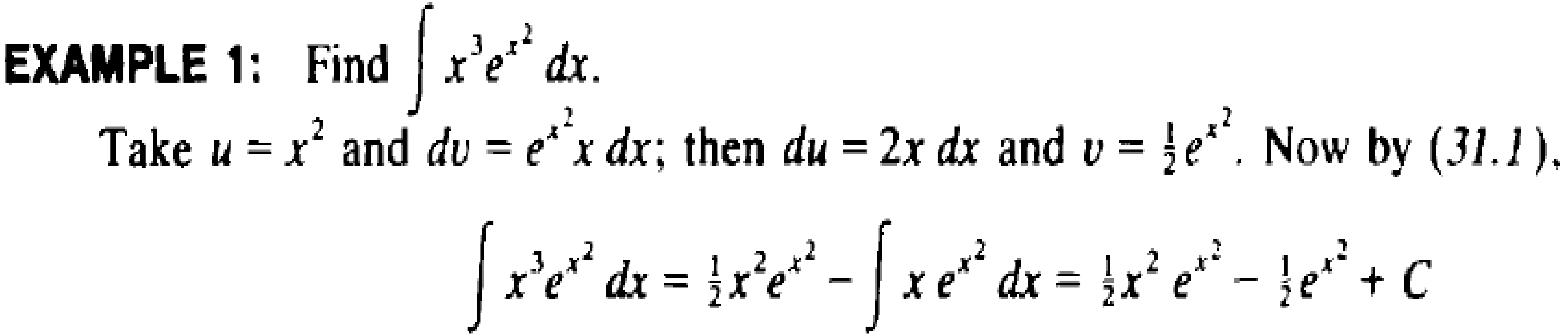
$$\begin{aligned}
\int u\,dv &= uv - \int v\,du && \text{Integration by parts formula} \\
\int x^2 \ln x\,dx &= \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx && \text{Substitute.} \\
&= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx && \text{Simplify.} \\
&= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. && \text{Integrate.}
\end{aligned}$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9} + C\right] = \frac{x^3}{3}\left(\frac{1}{x}\right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$

■

# INTEGRAL CALCULUS



**EXAMPLE 1:** Find $\int x^3 e^{x^2}\,dx$.

Take $u = x^2$ and $dv = e^{x^2} x\,dx$; then $du = 2x\,dx$ and $v = \frac{1}{2}e^{x^2}$. Now by (*31.1*),

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \int x\,e^{x^2}\,dx = \tfrac{1}{2}x^2\,e^{x^2} - \tfrac{1}{2}e^{x^2} + C$$

# INTEGRAL CALCULUS

**2.** Find $\int xe^x\,dx$.

Let $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

**3.** Find $\int x^2 \ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9}\,x^3 + C$$

# INTEGRAL CALCULUS

**4.** Find $\int x\sqrt{1+x}\,dx$.

Let $u = x$, $dv = \sqrt{1+x}\,dx$. Then $du = dx$, $v = \frac{2}{3}(1+x)^{3/2}$, and

$$\int x\sqrt{1+x}\,dx = \tfrac{2}{3}x(1+x)^{3/2} - \tfrac{2}{3}\int (1+x)^{3/2}\,dx = \tfrac{2}{3}x(1+x)^{3/2} - \tfrac{4}{15}(1+x)^{5/2} + C$$

**5.** Find $\int \arcsin x\,dx$.

Let $u = \arcsin x$, $dv = dx$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$, and

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C$$

# INTEGRAL CALCULUS

**6.** Find $\int \sin^2 x \, dx$.

Let $u = \sin x$, $dv = \sin x \, dx$. Then $du = \cos x \, dx$, $v = -\cos x$, and

$$\int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx = -\sin x \cos x + \int (1 - \sin^2 x) \, dx$$

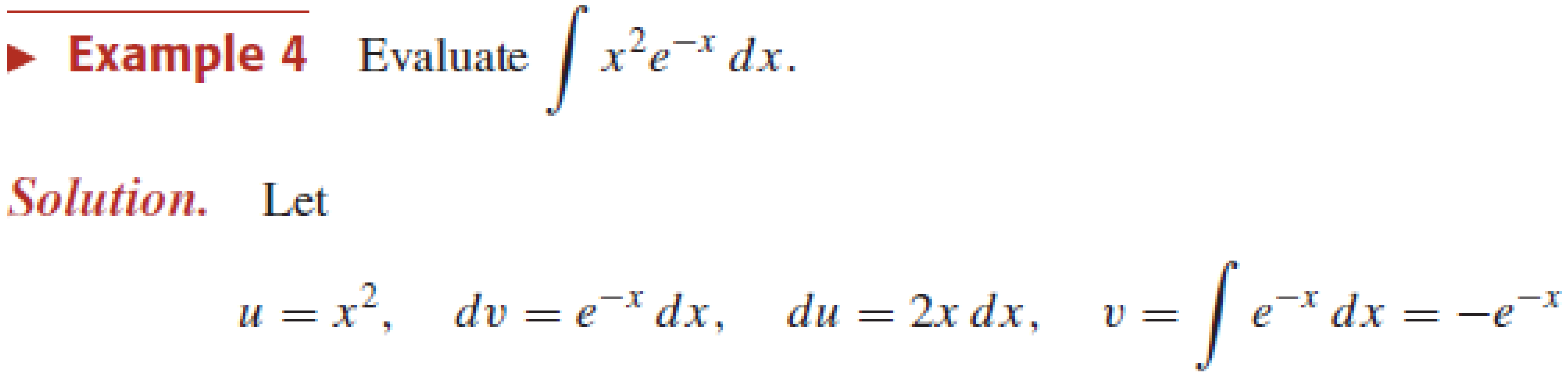
$$= -\tfrac{1}{2} \sin 2x + \int dx - \int \sin^2 x \, dx$$

Hence $\quad 2\int \sin^2 x \, dx = -\tfrac{1}{2} \sin 2x + x + C' \quad$ and $\quad \int \sin^2 x \, dx = \tfrac{1}{2} x - \tfrac{1}{4} \sin 2x + C$

# INTEGRAL CALCULUS

## REPEATED INTEGRATION BY PARTS

It is sometimes necessary to use integration by parts more than once in the same problem.



▶ **Example 4** Evaluate $\int x^2 e^{-x}\,dx$.

*Solution.* Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

# INTEGRAL CALCULUS

so that from (3)

$$\begin{aligned}\int x^2e^{-x}\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx \\ &= -x^2e^{-x} + 2\int xe^{-x}\,dx \end{aligned} \tag{4}$$

The last integral is similar to the original except that we have replaced $x^2$ by $x$. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

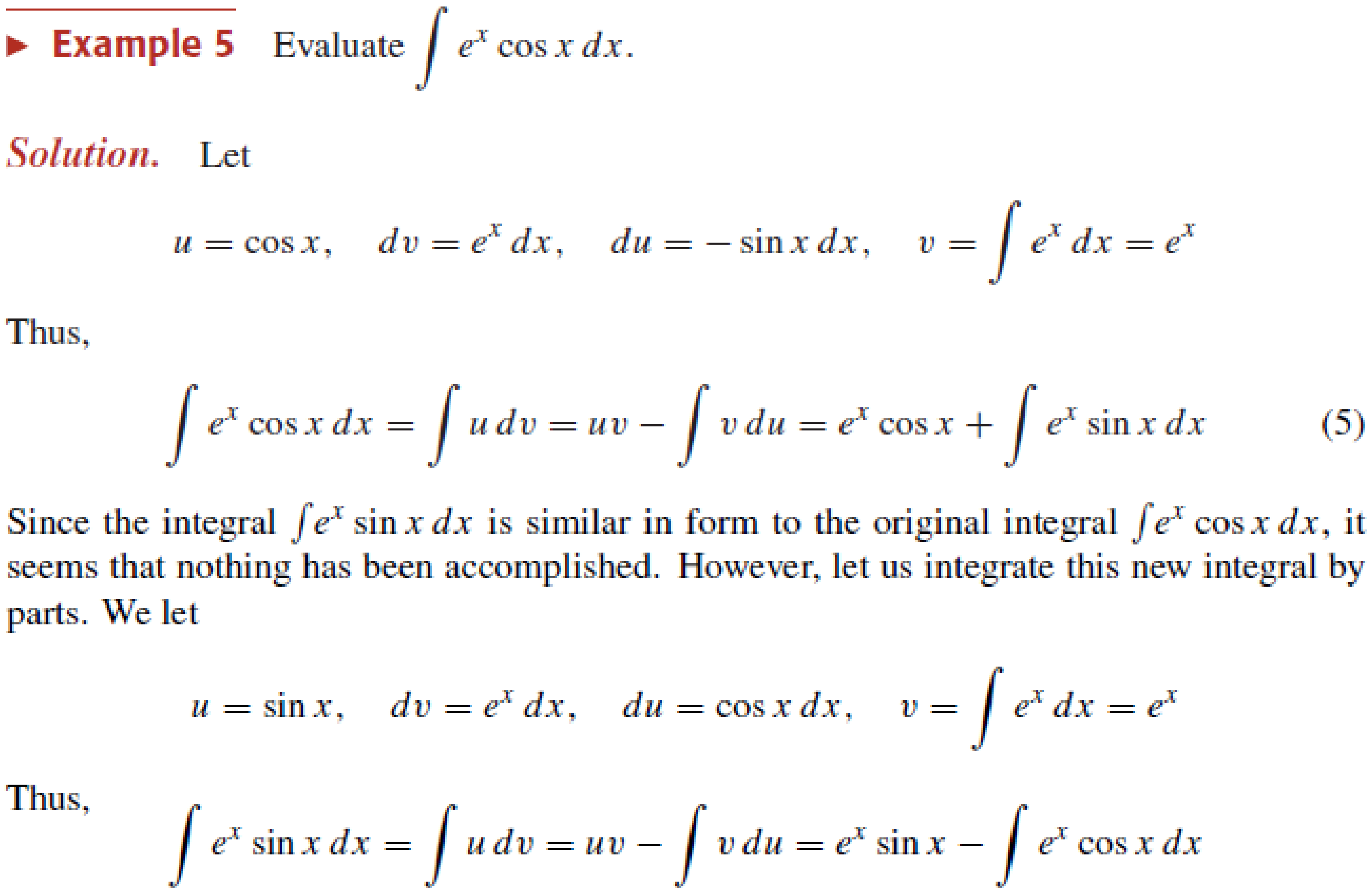
so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C \quad \blacktriangleleft\end{aligned}$$

# INTEGRAL CALCULUS



**▸ Example 5** Evaluate $\int e^x \cos x\, dx$.

*Solution.* Let

$$u = \cos x, \quad dv = e^x\, dx, \quad du = -\sin x\, dx, \quad v = \int e^x\, dx = e^x$$

Thus,

$$\int e^x \cos x\, dx = \int u\, dv = uv - \int v\, du = e^x \cos x + \int e^x \sin x\, dx \tag{5}$$

Since the integral $\int e^x \sin x\, dx$ is similar in form to the original integral $\int e^x \cos x\, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x\, dx, \quad du = \cos x\, dx, \quad v = \int e^x\, dx = e^x$$

Thus,

$$\int e^x \sin x\, dx = \int u\, dv = uv - \int v\, du = e^x \sin x - \int e^x \cos x\, dx$$

# INTEGRAL CALCULUS

Together with Equation (5) this yields

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx \tag{6}$$

which is an equation we can solve for the unknown integral. We obtain

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x \, dx = \tfrac{1}{2} e^x \cos x + \tfrac{1}{2} e^x \sin x + C \quad \blacktriangleleft$$

# INTEGRAL CALCULUS

**8.** Find $\int x^2 \sin x \, dx$.

Let $u = x^2$, $dv = \sin x \, dx$. Then $du = 2x \, dx$, $v = -\cos x$, and

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\int x \cos x \, dx$$

For the resulting integral, let $u = x$ and $dv = \cos x \, dx$. Then $du = dx$, $v = \sin x$, and

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\left(x \sin x - \int \sin x \, dx\right) = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

# INTEGRAL CALCULUS

**9.** Find $\int x^3 e^{2x}\, dx$.

Let $u = x^3$, $dv = e^{2x}\, dx$. Then $du = 3x^2\, dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x}\, dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\, dx$. Then $du = 2x\, dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2 e^{2x} - \int x e^{2x}\, dx\right) = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x e^{2x}\, dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\, dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left(\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\, dx\right) = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C$$

Find $\int x \sin x \, dx$.

$\underline{u} = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = \int dv$

$= \int \sin x \, dx$

$\underline{v} = -\cos x$

L I A T E

$uv - \int v \, du$

$= x(-\cos x) - \int -\cos x \, dx$

$= -x\cos x - (-\sin x) + c$

$= -x\cos x + \sin x + c$

3)

Evaluate $\int xe^x\,dx$.

L I $\underline{A}$ T $\underline{E}$

$u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = \int e^x dx = e^x$

$uv - \int v\,du$

$= xe^x - \int e^x dx$

$= \underline{xe^x - e^x + c}$ //

$e^x \underline{(x-1) + c}$ ✓

11) $\int x^3 \ln x \, dx$

$u = \ln x$ $\qquad dv = x^3 dx$

$du = \frac{dx}{x}$ $\qquad v = \frac{x^4}{4}$

$uv - \int v \, du$

$= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{dx}{x}$

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx$

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \frac{x^4}{4} + c$

$\boxed{= \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C}$ or $= \boxed{\frac{1}{4} x^4 \left(\ln x - \frac{1}{4}\right) + c}$ or $\boxed{\frac{1}{16} x^4 (4 \ln x - 1) + c}$

12) $\int (7-x) e^{x/2} dx$

$u = 7-x \qquad dv = e^{\frac{1}{2}x} dx$

$du = -dx \qquad v = 2e^{\frac{1}{2}x}$

$uv - \int v du$

$= (7-x) 2e^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x} - dx$

$= (14-2x)e^{\frac{1}{2}x} + 2\int e^{\frac{1}{2}x} dx$

$= (14-2x)e^{\frac{1}{2}x} + 4e^{1/2x} + c$

$= e^{\frac{1}{2}x}((14-2x)+4] + c$ or $\boxed{e^{\frac{1}{2}x}(18-2x)+c}$ or $\boxed{18e^{\frac{1}{2}x} - 2xe^{\frac{1}{2}x} + c}$

|   | D | I |
|---|---|---|
| + | $7-x$ | $e^{\frac{1}{2}x}$ |
| - | $-1$ | $2e^{\frac{1}{2}x}$ |
| + | $0$ | $4e^{1/2x}$ |

$$\int (2x+1) \sin 4x \, dx$$

$u = 2x+1 \qquad dv = \sin 4x \, dx$

$du = 2 \, dx \qquad v = -\frac{1}{4} \cos 4x$

$uv - v \, du$

$$= (2x+1)\left(-\frac{1}{4} \cos 4x\right) - \int -\frac{1}{4} \cos 4x (2 \, dx)$$

$$= -\frac{1}{4}(2x+1) \cos 4x + \frac{1}{2} \int \cos 4x \, dx$$

$$= -\frac{1}{4}(2x+1) \cos 4x + \frac{1}{2}\left(\frac{1}{4} \sin 4x\right) + c$$

$$\boxed{= -\frac{1}{4}(2x+1) \cos 4x + \frac{1}{8} \sin 4x + c}$$

$\int x\cos 4x\,dx$

$u = x \qquad dv = \cos 4x\,dx$

$du = dx \qquad v = \frac{1}{4}\sin 4x$

$uv - \int v\,du$

$= (x)(\frac{1}{4}\sin 4x) - \int \frac{1}{4}\sin 4x\,dx$

$= \frac{1}{4}x\sin 4x - \frac{1}{4}\int \sin 4x\,dx$

$= \frac{1}{4}x\sin 4x - \frac{1}{4}(-\frac{1}{4}\cos 4x) + c$

$\boxed{= \frac{1}{4}x\sin(4x) + \frac{1}{16}\cos 4x + c}$ or $\boxed{\frac{1}{16}(4x\sin(4x) + \cos(4x) + c}$

4) Evaluate $\int \ln x \, dx$.

$u = \ln x$

$du = \frac{dx}{x}$ or $\frac{1}{x} dx$

$dv = dx$

$v = \int dx = x$

$uv - \int v \, du$

$= \ln x (x) - \int x \frac{1}{x} dx$

$= x \ln x - \int dx$

$\underline{= x \ln x - x + C}$

Find $\int \underline{x}^3 e^{x^2}\,dx.$

$$\int x^2 \boxed{e^{x^2} x\,dx}$$

$u = x^2$

$du = 2x\,dx$

$dv = \underline{e^{x^2} x}\,dx$

$v = \int dv = \frac{1}{2}\int e^{x^2} x\,dx$

$V = \frac{1}{2} e^{x^2}$

$e^{x^2} = e^{x^2}\boxed{x\,dx}$

$e^{u} =$

$u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$uv - \int v\,du$

$= x^2 \frac{1}{2} e^{x^2} - \int \frac{1}{\cancel{2}} e^{x^2} \cancel{2}x\,dx$

$= \frac{1}{2} x^2 e^{x^2} - \int e^{x^2} x\,dx$

$= \underline{\frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C}$

Find $\int x^2 \ln x \, dx$.

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \int x^2 dx$

$v = \frac{x^3}{3}$

$uv - \int v \, du$

$= \ln x \, \frac{x^3}{3} - \int \frac{x^3}{3} \, \frac{dx}{x}$

$= \frac{x^3}{3} \ln x - \int \frac{x^2 dx}{3}$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \, \frac{x^3}{3} + c$

$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + c$

Find $\int x\sqrt{1+x}\,dx$.

$(1+x)$

$u = 1+x$

$du = dx$

$= 1\,dx = dx$

$= \int u^{1/2}\,du$

$= \frac{u^{3/2}}{3/2}$ or $\frac{2}{3}u^{3/2}$

$u = x \qquad dv = \sqrt{1+x}\,dx$

$du = dx \qquad = \int (1+x)^{1/2}\,dx$

$v = \frac{2}{3}(1+x)^{3/2}$

$uv - \int v\,du$

$\frac{u^{3/2+1}}{3/2+1} = \frac{u^{5/2}}{5/2}$

$= \frac{2}{5}u^{5/2}$

$$= x\left(\frac{2}{3}(1+x)^{3/2}\right) - \int \frac{2}{3}(1+x)^{3/2}\,dx$$

$$= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2}\,dx$$

$$= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\left(\frac{2}{5}(1+x)^{5/2}\right) + C$$

$$\boxed{= \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C}$$

Find $\int \arcsin x \, dx$.

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$uv - \int v \, du$

$= \arcsin x (x) - \int x \frac{1}{\sqrt{1-x^2}} dx$

$= x \arcsin x - -(1-x^2)^{1/2}$

$= x \arcsin x + (1-x^2)^{1/2} + C$

or

$x \arcsin x + \sqrt{1-x^2} + C$

$\int \frac{x \, dx}{\sqrt{1-x^2}}$

$= \int x(1-x^2)^{-1/2} dx$

$u = 1-x^2$

$du = -2x \, dx$

$dx = \frac{du}{-2x}$

$\int \cancel{x} u^{-1/2} \frac{du}{-2\cancel{x}}$

$= -\frac{1}{2} \int u^{-\frac{1}{2}} du$

$= -\frac{1}{2} \frac{u^{1/2}}{1/2} = -u^{1/2}$

$\boxed{= -(1-x^2)^{1/2}}$

Evaluate $\int \underline{x}^2 \underline{e^{-x}}\, \underline{dx}$.

$u = x^2 \qquad dv = e^{-x}dx$

$du = \underline{2x\,dx} \qquad v = \int e^{-x}dx$

$v = -e^{-x}$

$uv - \int v\,du$

$= -x^2 e^{-x} - \int -e^{-x}\, 2x\,dx$

$= -x^2 e^{-x} - 2\int -\underline{x e^{-x} dx}$

$= -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + C$

$= \boxed{-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C}$

$= \boxed{-e^{-x}(x^2 + 2x + 2) + C}$

LIATE

$\int x e^{-x} dx$

integration by parts

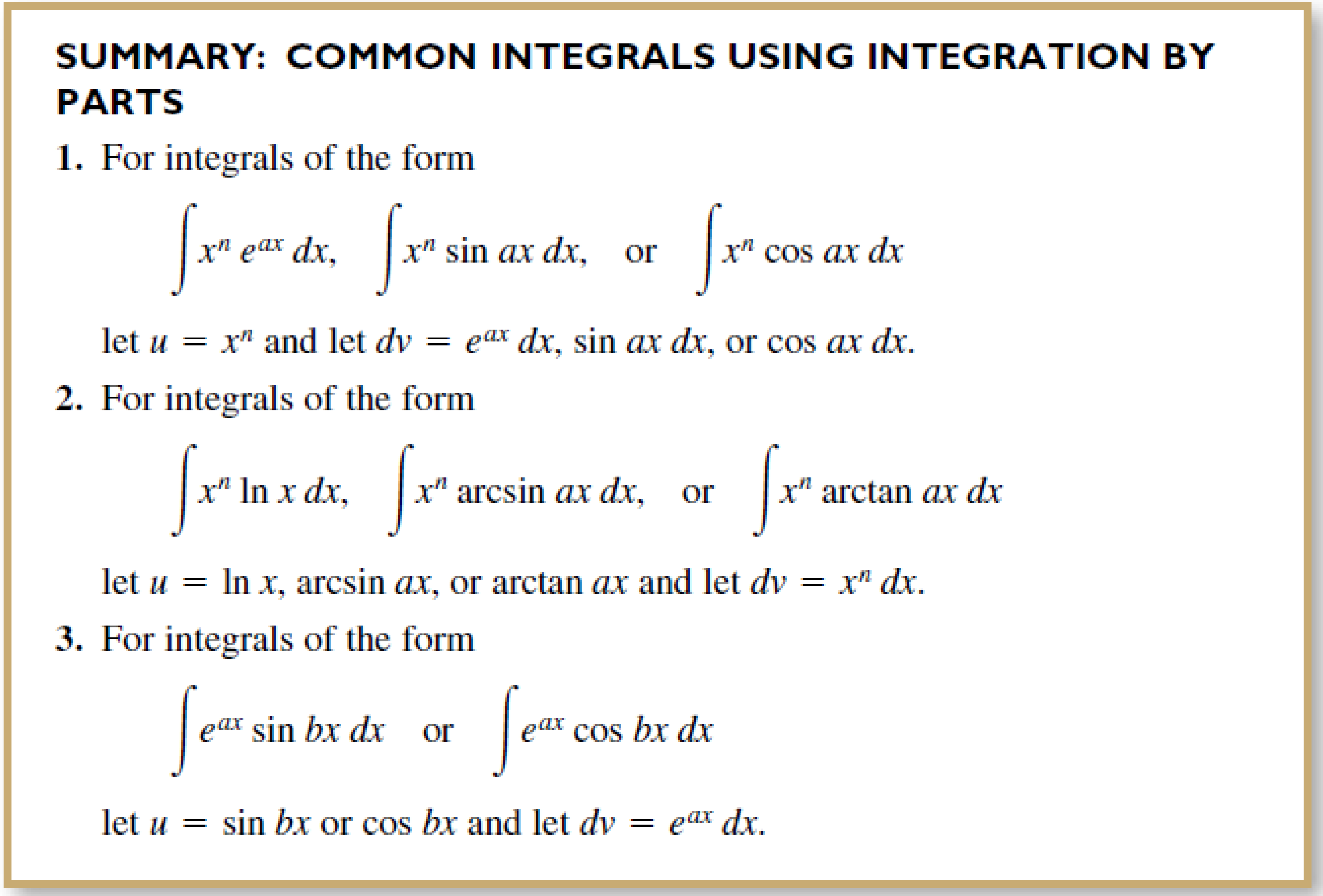
$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$uv - \int v\,du$

$= -xe^{-x} - \int -e^{-x} dx$

$= -xe^{-x} - e^{-x}$



## SUMMARY: COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax}\,dx, \quad \int x^n \sin ax\,dx, \quad \text{or} \quad \int x^n \cos ax\,dx$$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

2. For integrals of the form

$$\int x^n \ln x\,dx, \quad \int x^n \arcsin ax\,dx, \quad \text{or} \quad \int x^n \arctan ax\,dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

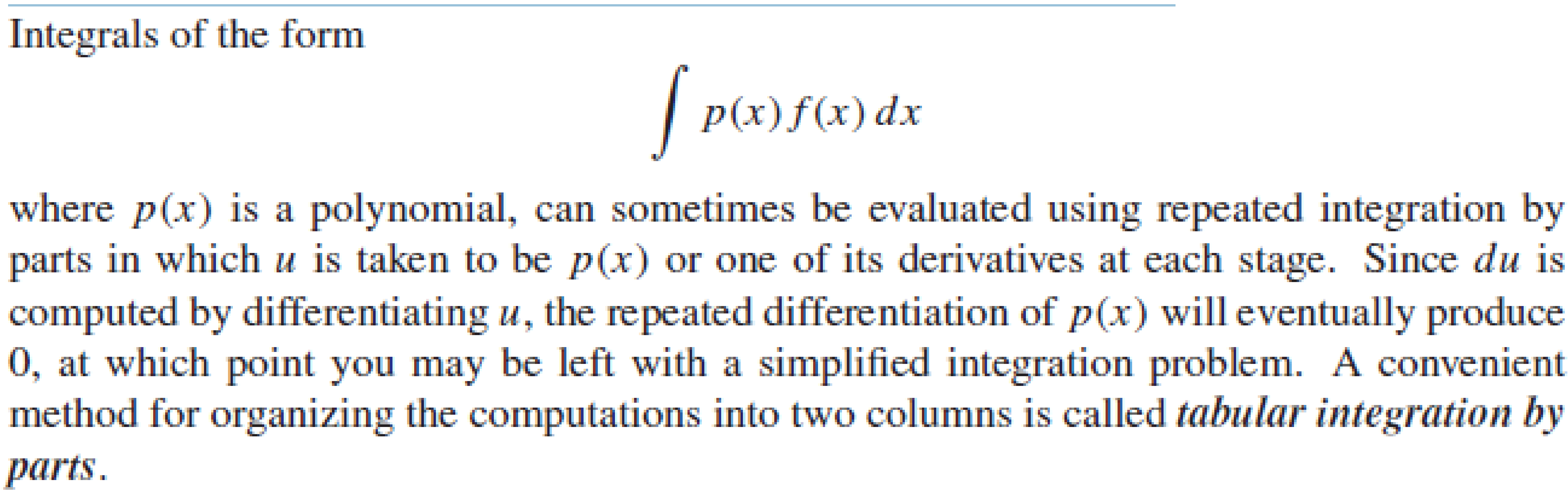
3. For integrals of the form

$$\int e^{ax} \sin bx\,dx \quad \text{or} \quad \int e^{ax} \cos bx\,dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

# INTEGRAL CALCULUS

## A TABULAR METHOD FOR REPEATED INTEGRATION BY PARTS

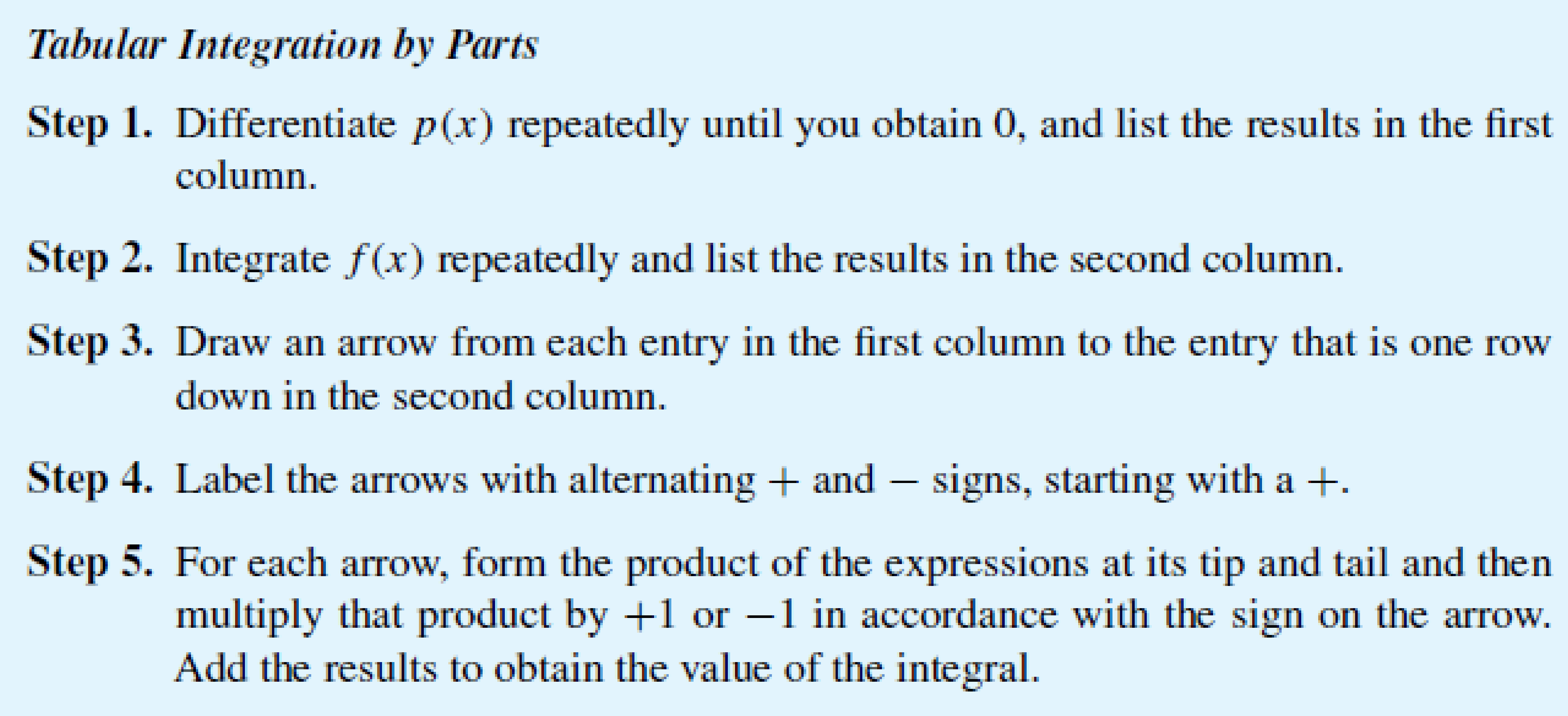


Integrals of the form

$$\int p(x)f(x)\,dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts in which $u$ is taken to be $p(x)$ or one of its derivatives at each stage. Since $du$ is computed by differentiating $u$, the repeated differentiation of $p(x)$ will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called *tabular integration by parts*.

# INTEGRAL CALCULUS



***Tabular Integration by Parts***

**Step 1.** Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

**Step 2.** Integrate $f(x)$ repeatedly and list the results in the second column.

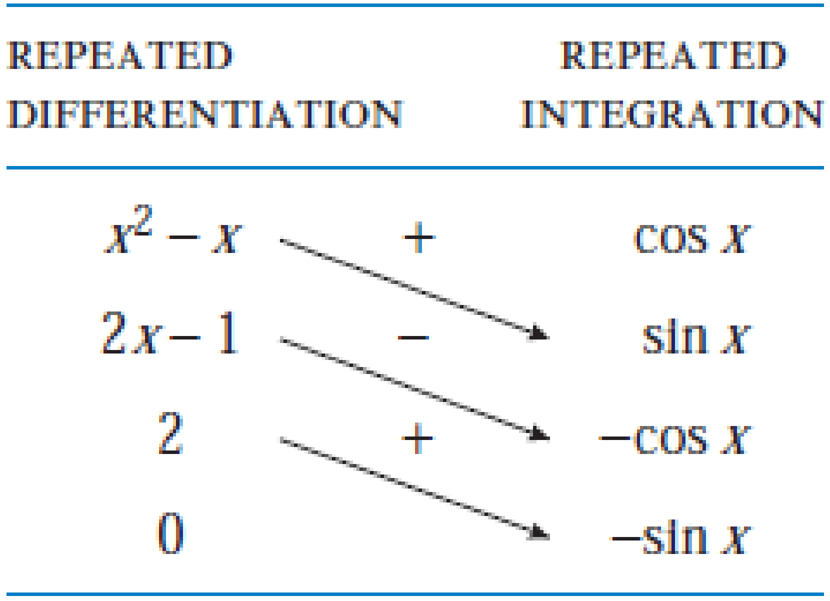
**Step 3.** Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

**Step 4.** Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

**Step 5.** For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or $-1$ in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

# INTEGRAL CALCULUS

This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x)\cos x\, dx$.



| REPEATED DIFFERENTIATION | | REPEATED INTEGRATION |
|---|---|---|
| $x^2 - x$ | $+$ | $\cos x$ |
| $2x - 1$ | $-$ | $\sin x$ |
| $2$ | $+$ | $-\cos x$ |
| $0$ | | $-\sin x$ |

$$\boxed{\begin{aligned}\int (x^2 - x)\cos x\, dx &= (x^2 - x)\sin x + (2x - 1)\cos x - 2\sin x + C\\ &= (x^2 - x - 2)\sin x + (2x - 1)\cos x + C\end{aligned}}$$

# INTEGRAL CALCULUS

$\int (x^2 - x)\cos x\,dx.$

Using Tabular or D I

| | Derivative | Integral |
|---|---|---|
| + | $x^2 - x$ | $\cos x$ |
| - | $2x - 1$ | $\sin x$ |
| + | $2$ | $-\cos x$ |
| - | $0$ | $-\sin x$ |

$$= (x^2 - x)\sin x + (2x-1)\cos x - 2\sin x + C$$

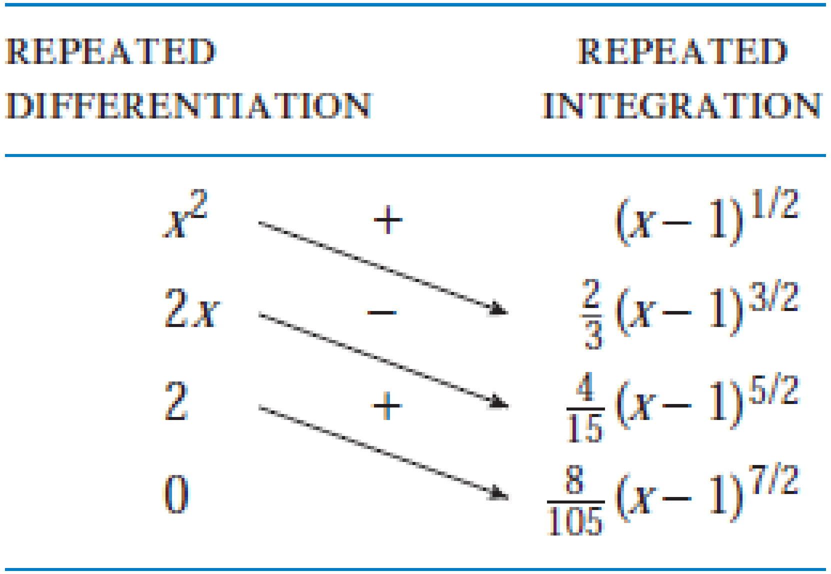
$$= x^2 \sin x - x\sin x - 2\sin x + (2x-1)\cos x + C$$

$$\boxed{= (x^2 - x - 2)\sin x + (2x-1)\cos x + C}$$

# INTEGRAL CALCULUS

**Example 6** $\int x^2\sqrt{x-1}\,dx$

*Solution.*



| REPEATED DIFFERENTIATION | | REPEATED INTEGRATION |
|---|---|---|
| $x^2$ | $+$ | $(x-1)^{1/2}$ |
| $2x$ | $-$ | $\frac{2}{3}(x-1)^{3/2}$ |
| $2$ | $+$ | $\frac{4}{15}(x-1)^{5/2}$ |
| $0$ | | $\frac{8}{105}(x-1)^{7/2}$ |

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C$$ ◄

# INTEGRAL CALCULUS

**Example 6** $\int x^2\sqrt{x-1}\,dx$

*Solution.*

| REPEATED DIFFERENTIATION | | REPEATED INTEGRATION |
|---|---|---|
| $x^2$ | $+$ | $(x-1)^{1/2}$ |
| $2x$ | $-$ | $\frac{2}{3}(x-1)^{3/2}$ |
| $2$ | $+$ | $\frac{4}{15}(x-1)^{5/2}$ |
| $0$ | | $\frac{8}{105}(x-1)^{7/2}$ |

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C \quad ◄$$

# INTEGRAL CALCULUS

**Ex 3**

$\int (3x^2 - x + 2)e^{-x}\,dx$. Then

| diff. |  | antidiff. |
|---|---|---|
| $3x^2 - x + 2$ |  | $e^{-x}$ |
|  | $\searrow +$ |  |
| $6x - 1$ |  | $-e^{-x}$ |
|  | $\searrow -$ |  |
| $6$ |  | $e^{-x}$ |
|  | $\searrow +$ |  |
| $0$ |  | $-e^{-x}$ |

$$I = \int (3x^2 - x + 2)e^{-x} = -(3x^2 - x + 2)e^{-x} - (6x - 1)e^{-x} - 6e^{-x} + C = -e^{-x}[3x^2 + 5x + 7] + C.$$

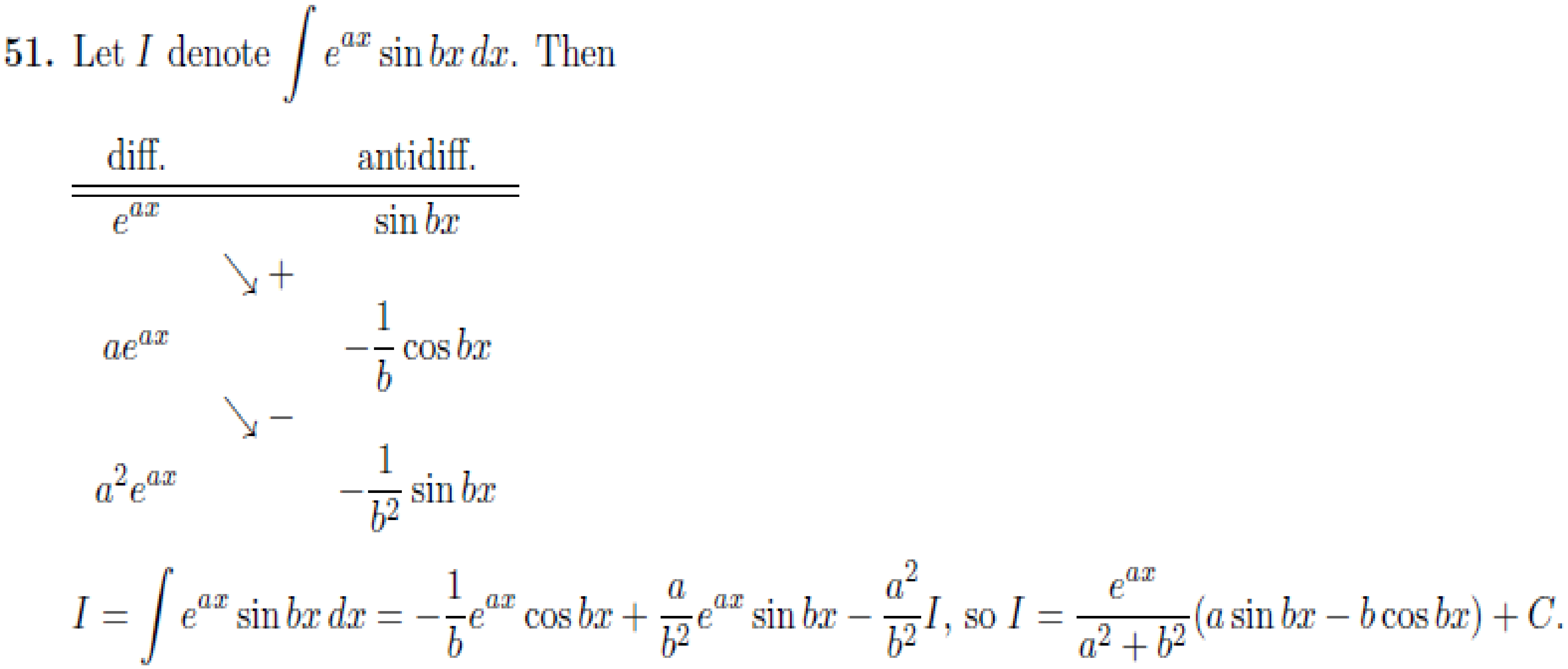
# INTEGRAL CALCULUS

Let $I$ denote $\int 4x^4 \sin 2x \, dx$. Then

$$\begin{array}{ccc}
\text{diff.} & & \text{antidiff.} \\
\hline\hline
4x^4 & & \sin 2x \\
 & \searrow + & \\
16x^3 & & -\frac{1}{2}\cos 2x \\
 & \searrow - & \\
48x^2 & & -\frac{1}{4}\sin 2x \\
 & \searrow + & \\
96x & & \frac{1}{8}\cos 2x \\
 & \searrow - & \\
96 & & \frac{1}{16}\sin 2x \\
 & \searrow + & \\
0 & & -\frac{1}{32}\cos 2x
\end{array}$$

$$I = \int 4x^4 \sin 2x \, dx = (-2x^4 + 6x^2 - 3)\cos 2x + (4x^3 - 6x)\sin 2x + C.$$

# INTEGRAL CALCULUS



**51.** Let $I$ denote $\int e^{ax}\sin bx\,dx$. Then

| diff. | | antidiff. |
|---|---|---|
| $e^{ax}$ | | $\sin bx$ |
| | $\searrow +$ | |
| $ae^{ax}$ | | $-\frac{1}{b}\cos bx$ |
| | $\searrow -$ | |
| $a^2e^{ax}$ | | $-\frac{1}{b^2}\sin bx$ |

$$I = \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I, \text{ so } I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

# INTEGRAL CALCULUS

# INTEGRAL CALCULUS

Integration by Parts (some cases)

1) $\int e^x \cos x \, dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x \, dx \qquad v = e^x$

$uv - \int v \, du$

$e^x \cos x - \int e^x(-\sin x) \, dx$

$e^x \cos x + \int e^x \sin x \, dx$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x \, dx \qquad v = e^x$

$e^x \cos x + e^x \sin x - \int \underbrace{e^x \cos dx}_{\text{Same as origina}}$

$$\underbrace{\int e^x \cos x \, dx}_{\text{same}} = e^x \cos x + e^x \sin x - \underset{\text{same}}{\int e^x \cos x \, dx}$$

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C$$

$$\boxed{\int e^x \cos x \, dx = \frac{e^x \cos x}{2} + \frac{e^x \sin x}{2} + C}$$

# INTEGRAL CALCULUS

Using DI method Case 2

$$\int e^x \cos x \, dx$$

$$\begin{array}{ccccc} & D & & I \\ + & \cos x & \overset{1}{\searrow} & e^x \\ - & -\sin x & \overset{2}{\searrow} & e^x \\ + & -\cos x & \underset{3}{\longleftrightarrow} & e^x \end{array}$$

$$e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x + c$$

$$\int e^x \cos x \, dx = \boxed{\frac{e^x \cos x}{2} + \frac{e^x \sin x}{2} + c}$$

# INTEGRAL CALCULUS

Ex 2 $\int e^{2x}\sin 3x\,dx$

| | D | I |
|---|---|---|
| + | $\sin 3x$ | $e^{2x}$ |
| - | $3\cos 3x$ | $\frac{1}{2}e^{2x}$ |
| + | $-9\sin 3x$ | $\frac{1}{4}e^{2x}$ |

Case 2 similar to given

$$\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \int \frac{9}{4}e^{2x}\sin 3x\,dx$$

$$\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int \underline{e^{2x}\sin 3x\,dx}$$

$$\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\,dx$$

$$\int e^{2x}\sin 3x\,dx + \frac{9}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x + C$$

$$\frac{13}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x + C$$

$$\int e^{2x}\sin 3x\,dx = \frac{4}{26}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C$$

$$\boxed{= \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C}$$

# INTEGRAL CALCULUS

DI Case 3

3) $\int \frac{\ln x}{\sqrt{x}} dx$

| | | |
|---|---|---|
| | D | I |
| + | $\ln x$ | $\frac{1}{\sqrt{x}}$ or $\frac{1}{x^{\frac{1}{2}}} \rightarrow x^{-\frac{1}{2}}$ |
| − | $\frac{1}{x}$ | $2x^{\frac{1}{2}}$ |

case 3 if integrable

$$2x^{\frac{1}{2}}\ln x - \int \frac{1}{x} 2x^{\frac{1}{2}}$$

$$2x^{\frac{1}{2}}\ln x - 2\int \frac{x^{\frac{1}{2}}}{x} dx$$

$$2x^{\frac{1}{2}}\ln x - 2\int x^{-\frac{1}{2}} dx$$

$$2x^{\frac{1}{2}}\ln x - 2\left[2x^{\frac{1}{2}}\right] + c$$

$$2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}} + c$$

$$\boxed{2\sqrt{x}\ln x - 4\sqrt{x} + c}$$ or

$$\boxed{2\sqrt{x}(\ln x - 2) + c}$$

# INTEGRAL CALCULUS

4) $\int x^2 \ln x \, dx$

$$\begin{tikzcd}
D & I \\
+ \ \ln x \arrow[dr] & x^2 \\
- \ \frac{1}{x} & \frac{x^3}{3}
\end{tikzcd}$$

Case 3

integrable

$$= \frac{x^3}{3}\ln x - \int \frac{1}{x}\,\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\,\frac{x^3}{3} + c$$

$$= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c$$

or

$$\boxed{\frac{1}{9}x^3(3\ln x - 1) + c}$$

# INTEGRAL CALCULUS

Ma

5) $\int \sin\sqrt{x}\, dx$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$\int \sin u \; 2u\,du$

$2\int u \sin u \, du$

|  | D | I |
|---|---|---|
| + | $u$ | $\sin u$ |
| − | $1$ | $-\cos u$ |
| + | $0$ | $-\sin u$ |

$2(-u\cos u + \sin u) + c$

$-2u\cos u + 2\sin u + c$

$-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + c$

6) $\int \ln(2+x^2)\,dx$

| | D | I |
|---|---|---|
| + | $\ln(2+x^2)$ | $1$ |
| − | $\frac{2x}{2+x^2}$ | $x$ |

Integrate

$$x\ln(2+x^2) - \int \frac{2x^2}{2+x^2}\,dx$$

$$\begin{array}{r|l} & 2 \\ 2+x^2 & 2x^2 \\ & -(2x^2+4) \\ \hline & -4 \end{array}$$

$$2 - \frac{4}{2+x^2}\,dx$$

$$-\int\left(2 - \frac{4}{2+x^2}\right)dx$$

$$-\int 2\,dx + \int \frac{4\,dx}{2+x^2}$$

$$-\int 2\,dx + 4\int \frac{dx}{2+x^2}$$

$a = \sqrt{2}$

$u = x$

$$-2x + 4\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right)$$

$$-2x + \frac{4}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}$$

$$\frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$\boxed{x\ln(2+x^2) - 2x + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}$$