CALCULUS 2

ANTIDERIVATIVES (INTEGRAL)

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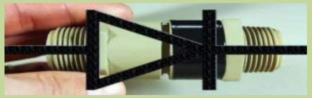
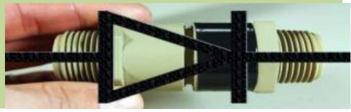


OBJECTIVES



- define and interpret definite integral,
- identify and distinguish the different properties of the definite integrals; and
- evaluate definite integrals

INTEGRATION OF ABSOLUTE VALUE FUNCTION

- one way to compute definite integrals of absolute values of functions, that is when the integrand has the form $|f(x)|$. We do this by dividing the domain up into intervals on which $f(x) \geq 0$ and intervals on which $f(x) \leq 0$. Then adding up the result

INTEGRATION OF ABSOLUTE VALUE FUNCTION

EXAMPLE

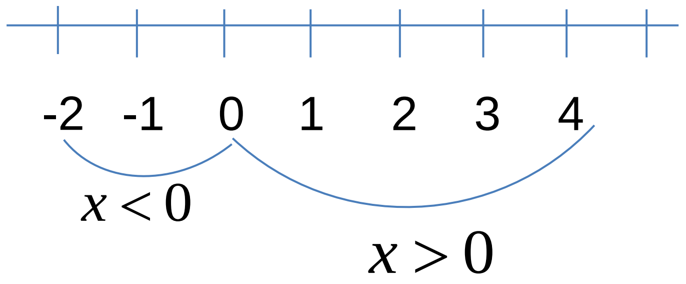
$$\text{Recall } |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

1. $\int_{-2}^{4} |x|\, dx$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\int_{-2}^{4} |x|\, dx = \int_{-2}^{0} -x dx + \int_{0}^{4} x dx$$

$$= -\frac{x^2}{2}\Big]_{-2}^{0} + \frac{x^2}{2}\Big]_{0}^{4} = 0 - \left[-\frac{(-2)^2}{2} \right] + \frac{(4)^2}{2} - 0$$



$$= 2 + 8 = 10$$

INTEGRATION OF ABSOLUTE VALUE FUNCTION

EXAMPLE

$$\int_0^3 |2x^2 - 8|dx.$$

On the interval $[0, 3]$, we have $2x^2 - 8 \leq 0$ when $x \in [0, 2]$ and $2x^2 - 8 \geq 0$ when $x \in [2, 3]$. Alternatively, we can write

$$|2x^2 - 8| = \begin{cases} 8 - 2x^2 & \text{for } x \in [0, 2] \\ 2x^2 - 8 & \text{for } x \in [2, 3] \end{cases}$$

INTEGRAL CALCULUS

$$\int_0^3 |2x^2 - 8|dx.$$

$$\begin{aligned}
\int_0^3 |2x^2 - 8|dx &= \int_0^2 8 - 2x^2 dx + \int_2^3 2x^2 - 8dx \\
&= [8x - 2x^3/3]_0^2 + [2x^3/3 - 8x]_2^3 \\
&= (16 - 16/3) + ((18 - 24) - (16/3 - 16)) \\
&= 26 - 32/3 = 46/3.
\end{aligned}$$

INTEGRAL CALCULUS

Find $\int_0^{10} |x-5|\,dx.$

$$|x-5| = \begin{cases} x-5, & 5, 10 \\ 5-x, & 0, 5 \end{cases}$$

$$\begin{aligned}
\int_0^{10} |x-5|\,dx &= \int_0^5 |x-5|\,dx + \int_5^{10} |x-5|\,dx \\
&= \int_0^5 5-x\,dx + \int_5^{10} x-5\,dx \\
&= \left[5x - \frac{1}{2}x^2\Big|_0^5\right] + \left[\frac{1}{2}x^2 - 5x\Big|_5^{10}\right] \\
&= \left[\left(5(5) - \frac{1}{2}(5)^2\right) - \left(5(0) - \frac{1}{2}(0)^2\right)\right] + \left[\left(\frac{1}{2}(10)^2 - 5(10)\right) - \left(\frac{1}{2}(5)^2 - 5(5)\right)\right] \\
&= 25
\end{aligned}$$

INTEGRAL CALCULUS

Steps to apply in general.

(1) Determine the values of x when the definition of $|f(x)|$ changes. Say they are $a < x_1 < x_2 < \cdots < x_n < b$.

(2) Check which (if any) of these values lie in the interval of integration $[a, b]$.

1

(3) Split the integral of $|f(x)|$ over $[a, b]$ into several pieces

$$\int_a^b |f(x)|\,dx = \int_a^{x_1} |f(x)|\,dx + \int_{x_1}^{x_2} |f(x)|\,dx + \cdots + \int_{x_n}^{b} |f(x)|\,dx.$$

Replace each $|f(x)|$ by either $f(x)$ or $-f(x)$, depending on the details of your particular problem.

INTEGRAL CALCULUS

) $\int_0^3 |x-2|\ dx$

) $\int_0^6 |2x-4|\ dx$

$\int_0^4 |x+3|\ dx$

Example 2. Set up the equation (2) for the absolute value integral

$$\int_{-2}^{5} |-t^2 + 6t - 8|\, dt.$$

Solution. For ease of reference call that function $v(t)$. To evaluate this integral we must understand the sign of $v(t) = -(t-4)(t-2)$ on the interval $[-2, 5]$. Because the roots of $v(t)$ are $t = 2, 4$ and the leading coefficient of $v(t)$ is negative, it follows that $v(t)$ is negative on $[-2, 2]$, positive on $[2, 4]$, and negative again on $[4, 5]$. You can graph this to see for yourself. This means the definition of $|v(t)|$ changes at $t_1 = 2, t_2 = 4, t_3 = 5$. Accordingly we must split up

$$[-2, 5] = [-2, 2] \cup [2, 4] \cup [2, 5]$$

and use the definitions according to equation (1)

$$|v(t)| = \begin{cases} -v(t), & -2 \le t \le 2 \\ v(t), & 2 \le t \le 4 \\ -v(t), & 4 \le t \le 5 \end{cases}$$

Therefore

$$\begin{aligned}\int_{-2}^{5} |v(t)|\,dt &= \int_{-2}^{2} -v(t)\,dt + \int_{2}^{4} v(t)\,dt + \int_{4}^{5} -v(t)\,dt \\ &= \int_{-2}^{2} t^2 - 6t + 8\,dt + \int_{2}^{4} -t^2 + 6t - 8\,dt + \int_{4}^{5} t^2 - 6t + 8\,dt\end{aligned}$$

We leave this unevaluated for the sake of neatness, but the reader should be able to use FTC2 and linearity/power rule to calculate these three definite integrals. The final answer is 40. □

INTEGRATION OF PIECEWISE FUNCTION

EXAMPLE

1. $\int_{-2}^{4} f(x)dx;\quad \mathrm{f(x)} = \begin{cases} 2+x^2, & -2 \le x < 0 \\ \frac{1}{2}x+2, & 0 \le x \le 4 \end{cases}$

solution

$$\int_{-2}^{4} f(x)dx = \int_{-2}^{0}\left(2+x^2\right)dx + \int_{0}^{4}\left(\frac{1}{2}x+2\right)dx$$

$$= 2x+\frac{x^3}{3}\Big]_{-2}^{0} + \frac{x^2}{4}+2x\Big]_{0}^{4} = \frac{56}{3}$$

INTEGRATION OF PIECEWISE FUNCTION

EXAMPLE

1. $\int_{-2}^{4} f(x)dx;\quad f(x) = \begin{cases} 2 + x^2, & -2 \le x < 0 \\ \frac{1}{2}x + 2, & 0 \le x \le 4 \end{cases}$

solution

$$\int_{-2}^{4} f(x)dx = \int_{-2}^{0} \left(2 + x^2\right)dx + \int_{0}^{4} \left(\frac{1}{2}x + 2\right)dx$$

$$= \left. 2x + \frac{x^3}{3} \right]_{-2}^{0} + \left. \frac{x^2}{4} + 2x \right]_{0}^{4} = \frac{56}{3}$$

INTEGRATION OF PIECEWISE FUNCTION

EXAMPLE

Evaluate $\int_0^3 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$

$$= \frac{x^3}{3}\Big]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}$$

EXAMPLE

$$6. \quad \int_{-2}^{1} (2t^2 - 1)^2 \, dt = \left(\frac{4t^5}{5} - \frac{4t^3}{3} + t \right) \Bigg|_{-2}^{1}$$

$$6. \quad \int_{-2}^{1} (2t^2 - 1)^2 \, dt = \left(\frac{4t^5}{5} - \frac{4t^3}{3} + t \right) \Bigg|_{-2}^{1} = \frac{7}{15} + \frac{254}{15} = \boxed{\frac{87}{5} = 17.4}$$

$$\int_{-1}^{1} (2t^2 - 1)^3 \, dt \qquad \int_{-1}^{1} (-1 + 2t^2)^3 \, dt$$

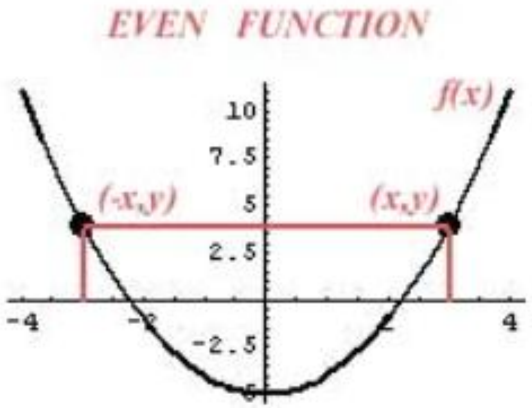
Integral even or odd function

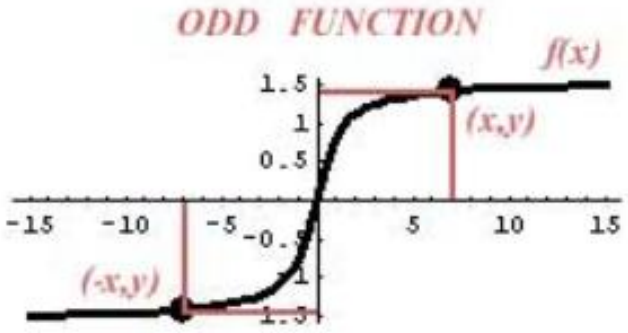
Sometimes we can simplify a definite integral if we recognize that the function we're integrating is an even function or an odd function. If the function is neither even nor odd, then we proceed with integration like normal.

To find out whether the function is even or odd, we'll substitute $-x$ into the function for x. If we get back the original function $f(x)$, the function is even. If we get back the original function multiplied by -1, the function is odd. In other words,

If $f(-x) = f(x)$, the function is even

If $f(-x) = -f(x)$, the function is odd





INTEGRAL CALCULUS

If we discover that the function is even or odd, the next step is to check the limits of integration (the interval over which we're integrating). In order to use the special even or odd function rules for definite integrals, our interval must be in the form $[-a, a]$. In other words, the limits of integration have the same number value but opposite signs, like $[-1, 1]$ or $[-5, 5]$.

The rules for integrating even and odd functions

If the function is even or odd and the interval is $[-a, a]$, we can apply these rules:

When $f(x)$ is even,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

When $f(x)$ is odd,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Example

Integrate.

$$\int_{-2}^{2} 3x^2 + 2\ dx$$

First we'll check to see if the function meets the criteria for an even or odd function. To see if it's even, we'll substitute $-x$ for x.

$$f(x) = 3x^2 + 2$$

$$f(-x) = 3(-x)^2 + 2$$

$$f(-x) = 3x^2 + 2$$

After substituting $-x$ for x, we were able to get back to the original function, which means we can say that

$$f(x) = f(-x)$$

and therefore that the function is even.

Looking at the given interval $[-2, 2]$, we see that it's in the form $[-a, a]$.

Since we know that our function is even and that our interval is symmetric about the y-axis, we can calculate our answer using the formula

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = 2\int_{0}^{2} 3x^2 + 2\ dx$$

Now, instead of integrating the left-hand side, we can instead integrate the right-hand side, and evaluating over the new interval will be a little easier.

$$\int_{-2}^{2} 3x^2 + 2\ dx = 2\left(\frac{3}{3}x^3 + 2x\right)\Big|_0^2$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = \left(2x^3 + 4x\right)\Big|_0^2$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = \left[2(2)^3 + 4(2)\right] - \left[2(0)^3 + 4(0)\right]$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = 24$$

Integrating odd functions

Example

Integrate.

$$\int_{-7}^{7} 3x^7 + 4\sin x \, dx$$

First we'll check to see if the function meets the criteria for an even or odd function. Let's start by testing it to see if it's an even function by substitutin $-x$ for x.

$$f(x) = 3x^7 + 4\sin x$$

$$f(-x) = 3(-x)^7 + 4\sin(-x)$$

$$f(-x) = -3x^7 - 4\sin x$$

$$f(x) \neq f(-x)$$

In order for the function to be even, $f(-x) = f(x)$. Since $f(x) \neq f(-x)$, this function is not even.

So we'll check to see if the function is odd. Remember that an odd function requires $f(-x) = -f(x)$. We can test this by substituting $-x$ for x.

$$f(x) = 3x^7 + 4\sin x$$

$$f(-x) = 3(-x)^7 + 4\sin(-x)$$

$$f(-x) = -3x^7 - 4\sin x$$

$$f(-x) = -\left(3x^7 + 4\sin x\right)$$

$$f(-x) = -f(x)$$

Because $f(-x)$ becomes $-f(x)$, we can say that the function is odd. Looking at the given interval $[-7, 7]$, we see that it's in the form $[-a, a]$.

Because $f(-x)$ becomes $-f(x)$, we can say that the function is odd. Looking at the given interval $[-7, 7]$, we see that it's in the form $[-a, a]$.

Since we know that our function is odd and that our interval is symmetric about the y-axis, we can calculate the answer using the formula

$$\int_{-a}^{a} f(x) \, dx = 0$$

$$\int_{-7}^{7} 3x^7 + 4\sin x \, dx = 0$$

EXAMPLE - Integrating Even and Odd Functions

Evaluate the following definite integrals

(a) $\int_{-2}^{2} x^2 dx$

(b) $\int_{-2}^{2} x^3 dx$

SOLUTION

(a) Since $f(x) = x^2$ is an even function, we can write

$$\int_{-2}^{2} x^2 dx = 2\int_{0}^{2} x^2 dx = 2\left[\frac{x^3}{3}\right]_0^2 =$$

$$= 2\left(\frac{8}{3} - 0\right) = \frac{16}{3}$$

(b) Since $f(x) = x^3$ is an odd function, we can write

$$\int_{-2}^{2} x^3 dx = 0$$

2

$$\int_{-1}^{1}(x^3 + 3x)\,dx$$

Solution :

Let $f(x) = x^3 + 3x$.

$$f(-x) = (-x)^3 + 3(-x)$$

$$f(-x) = -x^3 - 3x$$

$$f(-x) = -(x^3 + 3x)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

$$\int_{-1}^{1}(x^3 + 3x)\,dx = 0$$

3

$$\int_{-1}^{1}(\sin x \cdot \cos^4 x)\,dx$$

Solution :

Let $f(x) = \sin x \cdot \cos^4 x$.

$$f(-x) = \sin(-x) \cdot \cos^4(-x)$$

$$f(-x) = -\sin x \cdot [\cos(-x)]^4$$

$$f(-x) = -\sin x \cdot (\cos x)^4$$

$$f(-x) = -\sin x \cdot \cos^4 x$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

$$\int_{-1}^{1}(\sin x \cdot \cos^4 x)\,dx = 0$$

4 $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 \cdot cos^3 x\right) dx$$

Solution :

Let $f(x) = x^3 \cdot \cos^3 x$.

$$f(-x) = (-x)^3 \cdot \cos^3(-x)$$

$$f(-x) = -x^3 \cdot [\cos(-x)]^3$$

$$f(-x) = -x^3 \cdot (\cos x)^3$$

$$f(-x) = -x^3 \cdot \cos^3 x$$

$$f(-x) = -f(x)$$

f(x) is an odd function.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 \cdot cos^3 x\right) dx = 0$$

Example 5 :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(cos^3 x\right) dx$$

Solution :

Let $f(x) = \cos^3 x$.

$$f(-x) = \cos^3(-x)$$

$$f(-x) = [\cos(-x)]^3$$

$$f(-x) = (\cos x)^3$$

$$f(-x) = \cos^3 x$$

$$f(-x) = f(x)$$

f(x) is an even function.

Example 9 :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (sin^2 x)\, dx$$

Solution :

Let f(x) = sin^2x.

$$f(-x) = sin^2(-x)$$

$$f(-x) = [sin(-x)]^2$$

$$f(-x) = (-sinx)^2$$

$$f(-x) = sin^2x$$

$$f(-x) = f(x)$$

f(x) is an even function.

f(x) is an even function.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (sin^2 x)\, dx = 2\int_{0}^{\frac{\pi}{2}} (sin^2 x)\, dx$$

Cosine Double Angle Identity :

$$cos2x = cos^2x - sin^2x$$

$$cos2x = 1 - sin^2x - sin^2x$$

$$cos2x = 1 - 2sin^2x$$

$$2sin^2x = 1 - cos2x$$

$$sin^2x = \frac{1 - cos2x}{2}$$

$$= 2\int_{0}^{\frac{\pi}{2}} \left(\frac{1 - cos2x}{2}\right) dx$$

$$= 2 \cdot \frac{1}{2}\int_{0}^{\frac{\pi}{2}} (1 - cos2x) dx$$

$$= \int_{0}^{\frac{\pi}{2}} (1 - cos2x) dx$$

$$= \left(x - \frac{sin2x}{2}\right)_{0}^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - \frac{sin2\left(\frac{\pi}{2}\right)}{2}\right) - \left(0 - \frac{sin(0)}{2}\right)$$

$$= \left(\frac{\pi}{2} - \frac{sin(\pi)}{2}\right) - (0 - 0)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Example 11 :

$$\int_{-1}^{1} ln\left(\frac{3 - x}{3 + x}\right)dx$$

Solution :

$$Let\ f(x) = ln\left(\frac{3 - x}{3 + x}\right).$$

$$f(-x) = ln\left(\frac{3 - (-x)}{3 + (-x)}\right)$$

$$f(-x) = ln\left(\frac{3 + x)}{3 - x}\right)$$

$$f(-x) = ln(3 + x) - ln(3 - x)$$

$$f(-x) = -[-ln(3 + x) + ln(3 - x)]$$

$$f(-x) = -ln\left(\frac{3 - x)}{3 + x}\right)$$

f(-x) = -f(x)

f(x) is an odd function.

$$\int_{-1}^{1} ln\left(\frac{3 - x}{3 + x}\right) = 0$$