CALCULUS 2

ANTIDERIVATIVES (INTEGRAL)

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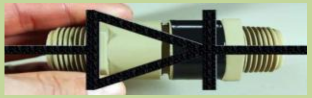
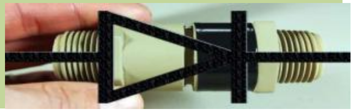


OBJECTIVES



- define and interpret definite integral,
- identify and distinguish the different properties of the definite integrals; and
- evaluate definite integrals

INTEGRAL CALCULUS

THE DEFINITE INTEGRAL

If *F(x)* is the integral of *f(x)dx*, that is, *F'(x) = f(x)dx* and if *a* and *b* are constants, then the definite integral is:

$$\int_a^b f(x)dx = F(x)\Big]_a^b$$

Fundamental Theorem of Calculus

$$= F(b) - F(a)$$

where a and b are called lower and upper limits of integration, respectively.

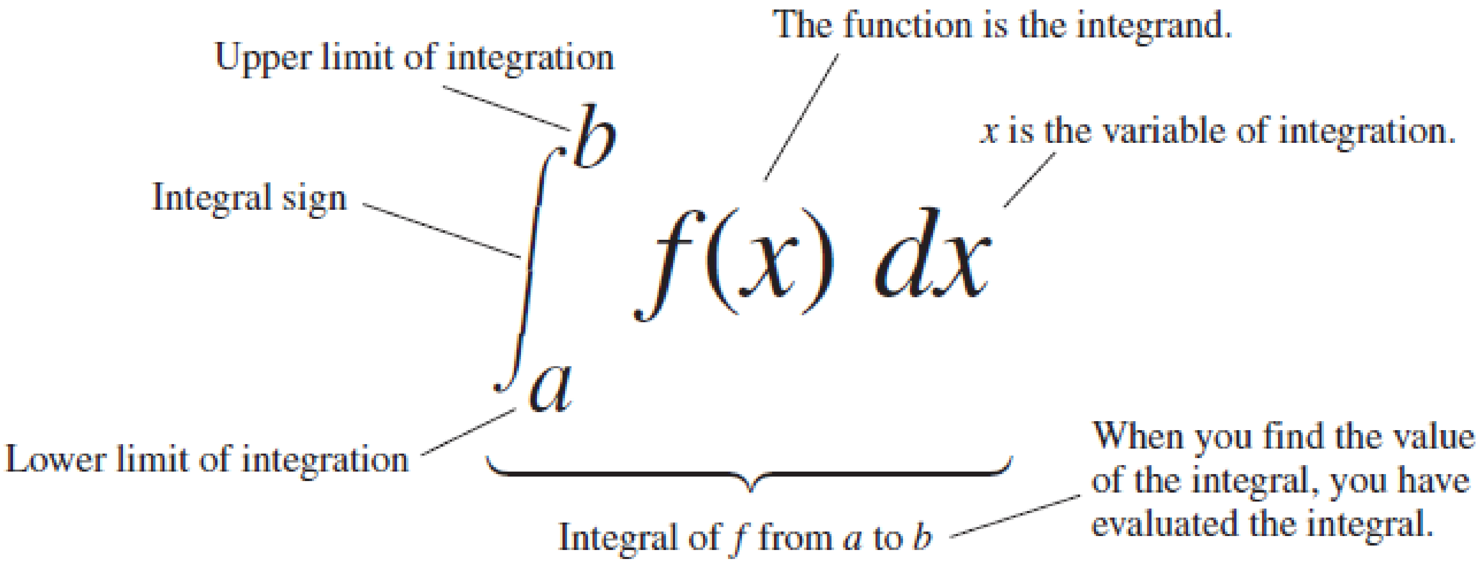
The definite integral link the concept of area to other important concepts such as length, volume, density, probability, and other work.

Notation and Existence of the Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



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PROPERTIES OF DEFINITE INTEGRAL

1. $If \quad a > b, \quad then$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$provided \quad f(x) \quad is \quad defined \quad in \quad the \quad closed \quad interval \quad [a,b]$

2. $If \quad a = b \quad and \quad F(x) \quad is \quad the \quad integral \quad of \quad f(x), \quad then$

$$\int_a^b f(x)dx \quad provided \quad f(a) \quad and \quad f(b) \quad exists.$$

$That \quad is,$

$$\int_a^b f(x)dx = \left[F(x) + C\right]_a^b = F(a) + C - \left[F(a) + C\right] = F(a) - F(a) = 0.$$

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3. $$\int_a^b \left[f_1(x) \pm f_2(x) \ldots + f_n(x) \right] dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx \ldots \pm f_n(x)dx$$

4. *If* $f(x)$ *is continuous function in the closed interval* $[a.b]$,

where $a < c < b$, *then,*

$$\int_a^b f(x)dx = \int_a^c f(x) + \int_c^b f(x)dx$$

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- To obtain the definite integral of a function, evaluate first its indefinite integral. Then applying the limits of integration, that is, substitute the upper limit of integration to all the variables contained in the indefinite integral, minus the function value of the indefinite integral using the lower limit of integration.

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TABLE 5.3 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	

EXAMPLE Using the Rules for Definite Integrals

Suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \qquad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ Rule 1

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$ Rules 3 and 4

$$= 2(5) + 3(7) = 31$$

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$ Rule 5

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① $\int_1^3 2dx = 2\int_1^3 x^0 dx = 2x\Big|_1^3$

$= (2 \times 3) - (2 \times 1) = 6 - 2$

$= 4.$

② $\int_{-2}^{1} (x^2 - 6x + 12)\, dx$

$= \frac{1}{3}x^3 - 3x^2 + 12x \Big]_{-2}^{1}$

$= \left[\frac{1}{3}(1)^3 - 3(1)^2 + 12(1)\right] - \left[\frac{1}{3}(-2)^3 - 3(-2)^2 + 12(-2)\right]$

$= \left[\frac{1}{3} - 3 + 12\right] - \left[-\frac{8}{3} - 12 - 24\right]$

$= 48$

▶ **Example 4**

$$\int_1^9 \sqrt{x}\,dx = \int x^{1/2}\,dx\Big]_1^9 = \frac{2}{3}x^{3/2}\Big]_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}$$ ◀

▶ **Example 5** Table 5.2.1 will be helpful for the following computations.

$$\int_4^9 x^2\sqrt{x}\,dx = \int_4^9 x^{5/2}\,dx = \frac{2}{7}x^{7/2}\Big]_4^9 = \frac{2}{7}(2187-128) = \frac{4118}{7} = 588\frac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx = -\frac{1}{5}\cos x\Big]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0-1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\,dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$\int_0^{\ln 3} 5e^x\,dx = 5e^x\Big]_0^{\ln 3} = 5[e^{\ln 3} - e^0] = 5[3-1] = 10$$

$$\int_{-e}^{-1} \frac{1}{x}\,dx = \ln|x|\Big]_{-e}^{-1} = \ln|-1| - \ln|-e| = 0 - 1 = -1$$

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x\Big]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$ ◀

Other examples definite integrals

$$\int \sin^2 x \cos x \, dx = \tfrac{1}{3}\sin^3 x$$

Evaluate $\int_0^{\pi/2} \sin^2 x \cos x \, dx$.

$$\int_0^{\pi/2} \sin^2 x \cos x \, dx = \tfrac{1}{3}\sin^3 x \Big]_0^{\pi/2}$$

$$= \left[\left(\sin\frac{\pi}{2}\right)^3 - (\sin 0)^3\right]$$

$$= \tfrac{1}{3}(1^3 - 0^3)$$

$$= \tfrac{1}{3}$$

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4) $\int \sec x \tan x = \sec x$

$\sec x \Big|_{-\frac{\pi}{4}}^{0}$

$= \sec 0 - \sec -\frac{\pi}{4}$

$= 1 - \sqrt{2}$

5) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + c$

$\frac{1}{3} \sin^3 x \Big|_0^{\frac{\pi}{2}}$

$= \left[\frac{1}{3} \sin^3 \frac{\pi}{2}\right] - \left[\frac{1}{3} \sin^3 0\right]$

$= \frac{1}{3} (1)^3 - \frac{1}{3} (0)$

$= \frac{1}{3}$

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6) $\int_0^{\frac{\pi}{2}} \cos x \sin x \, dx = \frac{1}{2}\sin^2 x + c$

$\left[\frac{1}{2}\sin^2 \frac{\pi}{2}\right] - \left[\frac{1}{2}\sin^2 0\right]$

$= \frac{1}{2}$

7) $\int \tan x \cdot \sec^2 x = \frac{1}{2}\tan^2 x + c$

$\frac{1}{2}\tan^2 x \Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\tan^2 \frac{\pi}{4} - \frac{1}{2}\tan^2 0$

$= \frac{1}{2}(1) - 0$

$= \frac{1}{2}$

(8) $\int_0^{\pi/2} \sqrt{\sin x + 1} \cos x \, dx.$

$$\int (\sin x+1)^{\frac{1}{2}} \cos x \, dx = \frac{2}{3}(\sin x+1)^{3/2} + C$$

$$\frac{2}{3}(\sin x+1)^{3/2} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3}\left(\sin \frac{\pi}{2}+1\right)^{3/2} - \frac{2}{3}(\sin 0+1)^{3/2}$$

$$= \frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(1)^{3/2} \qquad = \frac{2}{3}\sqrt{2^3} - \frac{2}{3}$$

$$= \frac{2}{3}\sqrt{8} - \frac{2}{3}$$

or

$$= \frac{2}{3}(\sqrt{8} - 1)$$

$$\frac{2}{3}(2\sqrt{2} - 1)$$

or $\frac{4\sqrt{2}}{3} - 2/3$

(9) $\int_0^{\pi/2} \frac{\sin x}{5}\,dx$ $= \int \frac{\sin x}{5} = -\frac{\cos x}{5} + c$

$= -\frac{1}{5}\cos\frac{\pi}{2} - \left(-\frac{1}{5}\cos 0\right)$

$= 0 + \frac{1}{5}$

$= \frac{1}{5}$ //

(10) $\int_0^{\pi/3} \sec^2 x\,dx$

$\tan x \Big|_0^{\frac{\pi}{3}}$

$= \tan\frac{\pi}{3} - \tan 0$

$= \sqrt{3} - 0$

$= \sqrt{3}$ //

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11) $\int 5e^x = 5e^x + c$

$5e^x \Big|_0^{\ln 3} = 5e^{\ln 3} - 5e^0$

$= 5(3) - 5(1)$

$= 15 - 5$

$= 10$

12) $\int \frac{1}{x} = \ln|x|$

$\ln|x| \Big|_{-e}^{-1} = \ln -1 - \ln -e$

$= 0 - 1$

$= -1$

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(13) $\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$

$u = (x^3+1)^{1/2}$

$du = 3x^2\,dx$

$\frac{1}{3}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{3}\frac{u^{3/2}}{3/2}$

$= \frac{2}{9}u^{3/2}\Big|_{-1}^{1}$

$= \frac{2}{9}(x^3+1)^{\frac{3}{2}}$

$= \frac{2}{9}\sqrt{(x^3+1)^3}\Big|_{-1}^{1}$

$= \frac{2}{9}\sqrt{8} - \frac{2}{9}(0)$

$= \frac{2}{9}\sqrt{8} = \frac{4}{9}\sqrt{2}$ //

(14) $\int_{0}^{2} x(x^2+1)^3\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\int u^3$

$= \frac{1}{2}\frac{u^4}{4} + c$

$= \frac{1}{8}u^4\Big|_0^2 = \frac{1}{8}(x^2+1)^4\Big|_0^2$

$= \frac{1}{8}(625) - \frac{1}{8}(0+1)^4$

$= 78$ //

INTEGRAL CALCULUS

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

TWO METHODS FOR MAKING SUBSTITUTIONS IN DEFINITE INTEGRALS

Method 1.

First evaluate the indefinite integral

$$\int f(g(x))g'(x)\,dx$$

by substitution, and then use the relationship

$$\int_a^b f(g(x))g'(x)\,dx = \left[\int f(g(x))g'(x)\,dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x-limits of integration.

Method 2.

Make the substitution (1) directly in the definite integral, and then use the relationship $u = g(x)$ to replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u = g(a)$ and $u = g(b)$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)\,du$$

that is expressed entirely in terms of u.

▶ **Example 1** Use the two methods above to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution by Method 1. If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_{x=0}^{2}$$

$$= \frac{(x^2+1)^4}{8}\Bigg]_{x=0}^{2} = \frac{625}{8} - \frac{1}{8} = 78$$

Solution by Method 2. If we make the substitution $u = x^2 + 1$ in (2), then

$$\text{if} \quad x = 0, \quad u = 1$$
$$\text{if} \quad x = 2, \quad u = 5$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du$$
$$= \left.\frac{u^4}{8}\right]_{u=1}^{5} = \frac{625}{8} - \frac{1}{8} = 78$$

which agrees with the result obtained by Method 1. ◄

INTEGRAL CALCULUS

5.9.1 THEOREM *If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 2 Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x \, dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9 \, dx$$

Solution (a). Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x \, dx \quad \left(\text{or } \tfrac{1}{2} du = \cos 2x \, dx\right)$$

With this substitution,

$$\text{if} \quad x = 0, \quad u = \sin(0) = 0$$
$$\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2}$$

so

$$\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx = \frac{1}{2}\int_0^{1/\sqrt{2}} u^5 \, du$$

$$= \frac{1}{2} \cdot \frac{u^6}{6}\Big]_{u=0}^{1/\sqrt{2}} = \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96}$$

Solution (b). Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x - 5$ unresolved in the integrand. However,

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u + 3) - 5 = 2u + 1$$

With this substitution,

$$\text{if} \quad x = 2, \quad u = 2 - 3 = -1$$
$$\text{if} \quad x = 5, \quad u = 5 - 3 = 2$$

so

$$\int_2^5 (2x - 5)(x - 3)^9\, dx = \int_{-1}^2 (2u + 1)u^9\, du = \int_{-1}^2 (2u^{10} + u^9)\, du$$
$$= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^2 = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right)$$
$$= \frac{52{,}233}{110} \approx 474.8 \blacktriangleleft$$

Example 3 Evaluate

$$\text{(a)} \int_0^{3/4} \frac{dx}{1-x} \qquad \text{(b)} \int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx$$

a) $u = 1 - x$

$du = -dx$

$x = 0 \quad u = 1 - 0 = 1$

$x = \frac{3}{4} \quad u = 1 - \frac{3}{4} = \frac{1}{4}$

$$= \int_1^{\frac{1}{4}} \frac{-du}{u}$$

$$= -\int_1^{\frac{1}{4}} \frac{du}{u}$$

$$= -\ln|u| \Big|_1^{\frac{1}{4}}$$

$$= -\left[\ln\tfrac{1}{4} - \ln 1\right] = \ln 4$$

INTEGRAL CALCULUS

$$\int_0^{\ln 3} e^x (1 + e^x)^{1/2}\, dx$$

$u = 1 + e^x$

$du = e^x dx$

$x = 0 \quad u = 1 + e^0 = 2$

$x = \ln \quad = u = 1 + e^{\ln 3}$

$= 1 + 3$

$= 4$

$$\int_2^4 u^{1/2}\, du$$

$$= \frac{2}{3} u^{3/2} \Big|_2^4$$

$$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(2)^{3/2}$$

$$= \frac{2}{3}\sqrt{4^3} - \frac{2}{3}\sqrt{2^3} = \frac{2}{3}\sqrt{64} - \frac{2}{3}\sqrt{8}$$

$$= \frac{16 - 4\sqrt{2}}{3}$$

: Evaluate $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

Put $\tan^{-1}x = t \qquad \therefore \quad \frac{1}{1+x^2}\,dx = dt$

When $x = 0$, $t = \tan^{-1}0 = 0$ ($\because \quad \tan 0 = 0$)

and when $x = 1$, $t = \tan^{-1}1 = \frac{\pi}{4} \quad \left(\because \quad \tan\frac{\pi}{4} = 1\right)$

$$\therefore \quad \int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx = \int_0^{\pi/4} t\,dt = \left[\frac{t^2}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[t^2\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0^2\right] = \frac{\pi^2}{32} \quad \text{Ans.}$$