# CALCULUS 1

## DERIVATIVES

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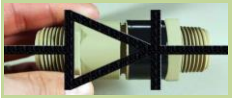
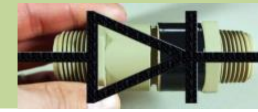


# OBJECTIVES



- Discuss chain rule
- Discuss higher derivative function

# CHAIN RULE

**2.6.1 THEOREM (The Chain Rule)** *If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at $x$. Moreover, if*

$$y = f(g(x)) \quad \textit{and} \quad u = g(x)$$

*then $y = f(u)$ and*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (1)$$

**AN ALTERNATIVE VERSION OF THE CHAIN RULE**

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x) \qquad (2)$$

A convenient way to remember this formula is to call *f* the "outside function" and *g* the "inside function" in the composition *f(g(x))* and then express (2) in words as:

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*The derivative of f(g(x)) is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.*

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{} \cdot \underbrace{g'(x)}_{}$$

Derivative of the outside function evaluated at the inside function

Derivative of the inside function

# DERIVATIVES

## GENERALIZED DERIVATIVE FORMULAS

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \qquad (3)$$

**Power Chain Rule,**

$$\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$$

**Example** Find

$$\frac{d}{dx}[x^2 - x + 2]^{3/4}$$

# DERIVATIVES

**Example 1** Find

$$\frac{d}{dx}[x^2 - x + 2]^{3/4}$$

***Solution***

Taking $u = x^2 - x + 2$ in the generalized derivative formula for $u^{3/4}$ yields

$$\frac{d}{dx}[x^2 - x + 2]^{3/4} = \frac{d}{dx}[u^{3/4}] = \frac{3}{4}u^{-1/4}\frac{du}{dx}$$

$$= \frac{3}{4}(x^2 - x + 2)^{-1/4} \cdot \frac{d}{dx}[x^2 - x + 2]$$

$$= \frac{3}{4}(x^2 - x + 2)^{-1/4}(2x - 1)$$

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## Example 2

$$y = (4x^2 - 2x + 5)^3$$

Let $y = u^3$ and $u = 4x^2 - 2x + 5$.

$$\frac{d}{dx}(u^3) = 3\,u^2$$

$$\frac{d}{dx}(4x^2\text{-}2\text{x}+5) = 8\text{x}-2$$

$$\frac{dy}{dx} = 3\,(4x^2 - 2\text{x}+5)^2\,(8\text{x}-2)$$

$$\frac{dy}{dx} = (24\text{x}-6)\,(4x^2 - 2\text{x}+5)^2$$

# DERIVATIVES

## Example 3

Differentiate $s = (t^2 - 3)^4$.

$$\frac{ds}{dt} = 4(t^2 - 3)^3(2t):$$

$$\frac{ds}{dt} = 8t(t^2 - 3)^3.$$

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$$y = (4x^2 - 3x + 1)^4$$

Find $y'$

$$y' = \frac{d}{dx}(4x^2 - 3x + 1)^4$$

$$= 4(4x^2 - 3x + 1)^3 \frac{d}{dx}(4x^2 - 3x + 1)$$

$$= 4(4x^2 - 3x + 1)^3 (8x - 3)$$

$$= 32x - 12 (4x^2 - 3x + 1)^3$$

$$u = 4x^2 - 3x + 1$$

$$du = 8x - 3$$

$$y = u^4$$

$$y' = 4u^3 \, du$$

$$= 4(4x^2 - 3x + 1)^3 (8x - 3)$$

$$= (32x - 12)(4x^2 - 3x + 1)^3$$

# Example 4

Differentiate $y = (x^2 + 4)^2(2x^3 - 1)^3$.

# Solution

**Use the Product Rule and the Power Chain Rule:**

$$y' = (x^2+4)^2 \frac{d}{dx}(2x^3-1)^3 + (2x^3-1)^3 \frac{d}{dx}(x^2+4)^2$$

$$= (x^2+4)^2(3)(2x^3-1)^2 \frac{d}{dx}(2x^3-1) + (2x^3-1)^3(2)(x^2+4)\frac{d}{dx}(x^2+4)$$

$$= (x^2+4)^2(3)(2x^3-1)^2(6x^2) + (2x^3-1)^3(2)(x^2+4)(2x)$$

$$= 2x(x^2+4)(2x^3-1)^2(13x^3+36x-2)$$

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**Example 4**

Differentiate $y = (x^2 + 4)^2(2x^3 - 1)^3$.

$$y' = (x^2+4)^2 \frac{d}{dx}(2x^3-1)^3 + (2x^3-1)^3 \frac{d}{dx}(x^2+4)^2$$

$$= (x^2+4)^2 \left[3(2x^3-1)^2 \frac{d}{dx}(2x^3-1)\right] + (2x^3-1)^3 \left[2(x^2+4)^1 \frac{d}{dx}(x^2+4)\right]$$

$$= (x^2+4)^2 \left[3(2x^3-1)^2 (6x^2)\right] + (2x^3-1)^3 \left[2(x^2+4)(2x)\right]$$

$$= (x^2+4)^2 \left[18x^2 (2x^3-1)^2\right] + (2x^3-1)^3 \left[4x(x^2+4)\right]$$

$$= \underbrace{(x^2+4)^2 (18x^2)(2x^3-1)^2}_{\text{common}} + \underbrace{(x^2+4)(4x)(2x^3-1)^3}_{\text{common}}$$

$$= (x^2+4)(2x)(2x^3-1)^2 \left[(x^2+4)9x + 2(2x^3-1)\right]$$

$$= (x^2+4)(2x)(2x^3-1)^2 \left[9x^3+36x+4x^3-2\right]$$

$$\boxed{= (x^2+4)(2x)(2x^3-1)^2(13x^3+36x-2)}$$

## Other examples. Differentiate the following

(a) $z = \dfrac{3}{(a^2 - y^2)^2}$ :     a is constant

(b) $f(x) = \sqrt{x^2 + 6x + 3}$ =

(c) $y = \dfrac{x^2}{\sqrt{4 - x^2}}$

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$$z = \frac{3}{(a^2 - y^2)^2}:$$

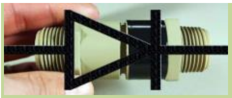
$$z = 3(a^2 - y^2)^{-2}$$

$$z' = 3\left[\frac{d}{dy}(a^2 - y^2)^{-2}\right]$$

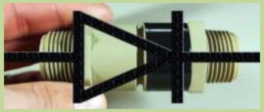
$$= 3\left[-2(a^2 - y^2)^{-3}\frac{d}{dy}(a^2 - y^2)\right.$$

$$= 3\left[-2(a^2 - y^2)^{-3}(-2y)\right.$$

$$z' = 12y(a^2 - y^2)^{-3} \quad \text{or} \quad \frac{12y}{(a^2 - y^2)^3}$$



# DERIVATIVES



) $f(x)=\sqrt{x^2+6x+3}$ =

$$f(x) = (x^2+6x+3)^{1/2}$$

$$f'(x) = \frac{d}{dx}(x^2+6x+3)^{1/2}$$

$$= \frac{1}{2}(x^2+6x+3)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+6x+3)$$

$$= \frac{1}{2}(x^2+6x+3)^{-1/2}(2x+6)$$

$$= \frac{1}{2}(2x+6)(x^2+6x+3)^{-\frac{1}{2}}$$

$$= (x+3)(x^2+6x+3)^{-1/2}$$ or $$\frac{x+3}{(x^2+6x+3)^{1/2}}$$ or $$\frac{x+3}{\sqrt{x^2+6x+3}}$$

# DERIVATIVES

c) $y = \frac{x^2}{\sqrt{4-x^2}} = \frac{x^2}{(4-x^2)^{1/2}}$

$y = x^2(4-x^2)^{-1/2}$

$y' = x^2 \frac{d}{dx}(4-x^2)^{-1/2} + (4-x^2)^{-1/2}\frac{d}{dx}(x^2)$

$= x^2\left[-\frac{1}{2}(4-x^2)^{-3/2}(-2x)\right] + (4-x^2)^{-1/2}(2x)$

$= x^3(4-x^2)^{-3/2} + 2x(4-x^2)^{-1/2}$

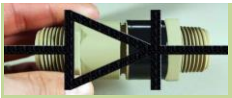
$= \frac{x^3}{(4-x^2)^{3/2}} + \frac{2x}{(4-x^2)^{1/2}}$

get the LCD

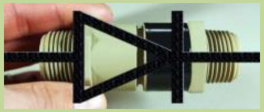
$= \frac{x^3 + 2x(4-x^2)}{(4-x^2)^{3/2}}$

$\frac{x^3 + 8x - 2x^3}{(4-x^2)^{3/2}}$

$= \frac{-x^3 + 8x}{(4-x^2)^{3/2}}$ or $\frac{x(-x^2+8)}{(4-x^2)^{3/2}}$



# DERIVATIVES

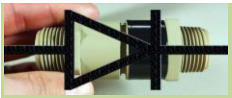


## Higher Derivatives

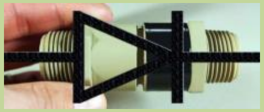
If $y = f(x)$ is differentiable, its derivative $y'$ is also called the *first derivative* of $f$. If $y'$ is differentiable, its derivative is called the *second derivative* of $f$. If this second derivative is differentiable, then its derivative is called the *third derivative* of $f$, and so on.

## Notation

| | | | | |
|---|---|---|---|---|
| First derivative: | $y'$, | $f'(x)$, | $\frac{dy}{dx}$, | $D_x y$ |
| Second derivative: | $y''$, | $f''(x)$, | $\frac{d^2y}{dx^2}$, | $D_x^2 y$ |
| Third derivative: | $y'''$, | $f'''(x)$, | $\frac{d^3y}{dx^3}$, | $D_x^3 y$ |
| $n^{\text{th}}$ derivative: | $y^{(n)}$, | $f^{(n)}$, | $\frac{d^ny}{dx^n}$, | $D_x^n y$ |

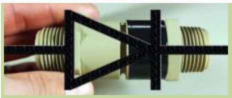


# DERIVATIVES

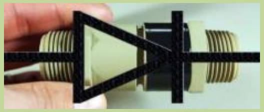


## HIGHER DERIVATIVES

The derivative *f'* of a function *f* is itself a function and hence may have a derivative of its own. If *f'* is differentiable, then its derivative is denoted by *f''* and is called the ***second derivative*** of *f* . As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of f . These successive derivatives are denoted by $f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$



# DERIVATIVES



## Notation

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| Second derivative: | $y''$, | $f''(x)$, | $\frac{d^2y}{dx^2}$, | $D_x^2 y$ |
| Third derivative: | $y'''$, | $f'''(x)$, | $\frac{d^3y}{dx^3}$, | $D_x^3 y$ |
| $n^{\text{th}}$ derivative: | $y^{(n)}$, | $f^{(n)}$, | $\frac{d^ny}{dx^n}$, | $D_x^n y$ |

## EXAMPLE 1

Find the first four derivatives of
$y = x^3 - 3x^2$

Solution

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

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Given $y = 3x^4$

Find the 5th derivative $y^{(5)}$

$$y = 3x^4$$

$$y' = \frac{d}{dx} 3x^4$$

1st $y' = 12x^3$

$$y'' = \frac{d}{dx}(12x^3)$$

2nd $y'' = 36x^2$

3rd $y''' = \frac{d}{dx}(36x^2)$

$$= 72x$$

4th $y^{(4)} = \frac{d}{dx}(72x)$

$$y^{(4)} = 72$$

5th $y^{(5)} = \frac{d}{dx}(72) = 0$

# DERIVATIVES

**EXAMPLE 2**

Find the third derivative (y''') of $y = \frac{1}{3+x}$

Solution:

$$y = (3+x)^{-1}$$

$$y' = -(3+x)^{-2}$$

$$y'' = 2(3+x)^{-3}$$

$$y''' = -6(3+x)^{-4} \quad \text{or} \quad = -\frac{6}{(3+x)^4}$$

**EXAMPLE 3**

At what order of derivative the function

f(x) = $2x^7 + x^8$ becomes 0?

# DERIVATIVES

$f(x) = 2x^7 + x^8$ becomes 0?

Solution

$f'(x) = 14x^6 + 8x^7$

$f''(x) = 84x^5 + 56x^6$

$f'''(x) = 420x^4 + 336x^5$

$f^{(4)}(x) = 1680x^3 + 1680x^4$

$f^{(5)}(x) = 5040x^2 + 6720x^3$

$f^{(6)}(x) = 10080x + 20160x^2$

$f^{(7)}(x) = 10080 + 40320x$

$f^{(8)}x = 0 + 40320$

$f^{(9)} = 0$

af $9^{th}$ derivative f(x) becomes 0

# REFERENCE


1. CALCULUS by H. Anton, et al 10th edition
2. Schaum's outline series CALCULUS 6th edition by Ayres/ Mendelson