CALCULUS 1

DERIVATIVES

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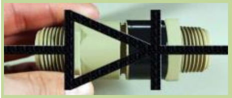
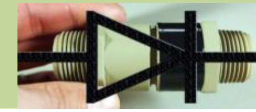


OBJECTIVES



- Discuss the derivative function

DERIVATIVES

THE DERIVATIVE FUNCTION

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

is called the *derivative of f with respect to x*. The domain of f' consists of all x in the domain of f for which the limit exists.

The term "derivative" is used because the function f' is *derived* from the function f by a limiting process.

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Example 1

Find the derivative with respect to *x* of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

Solution. It follows from (2) that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} (2x + h) = 2x$$

Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$. Since $y = 4$ if $x = 2$, the point-slope form of the tangent line is

$$y - 4 = 4(x - 2)$$

which we can rewrite in slope-intercept form as $y = 4x - 4$ (Figure 2.2.1). ◀

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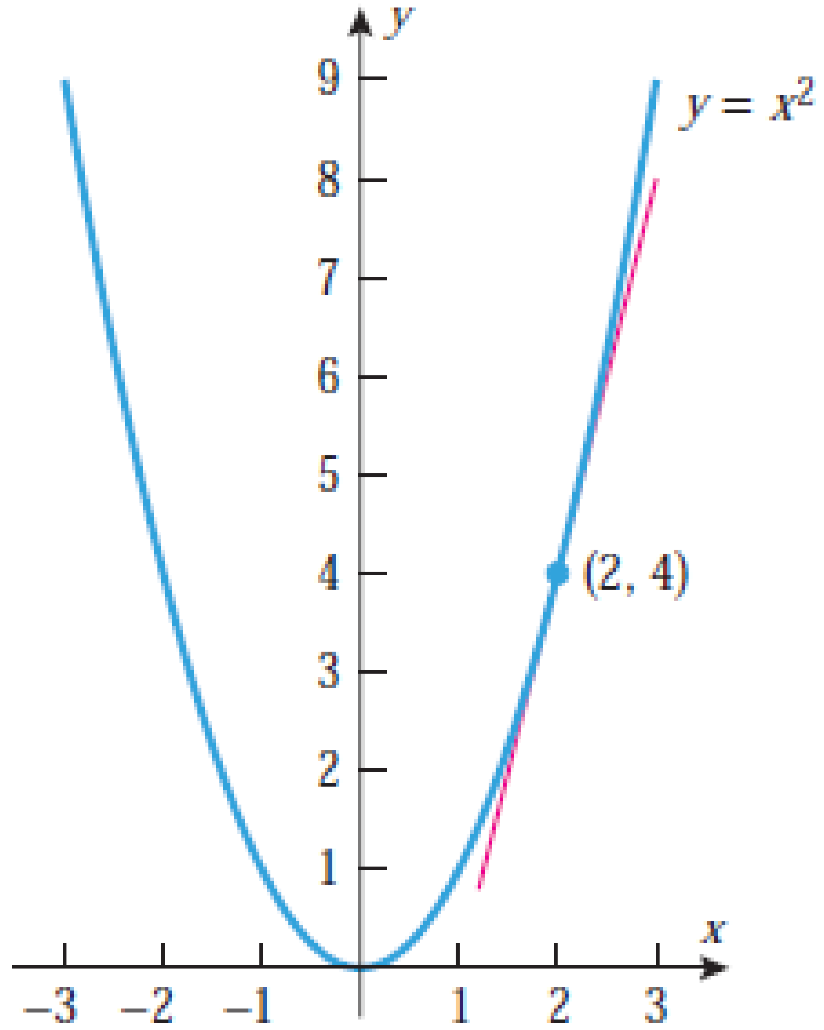

▲ **Figure 2.2.1**

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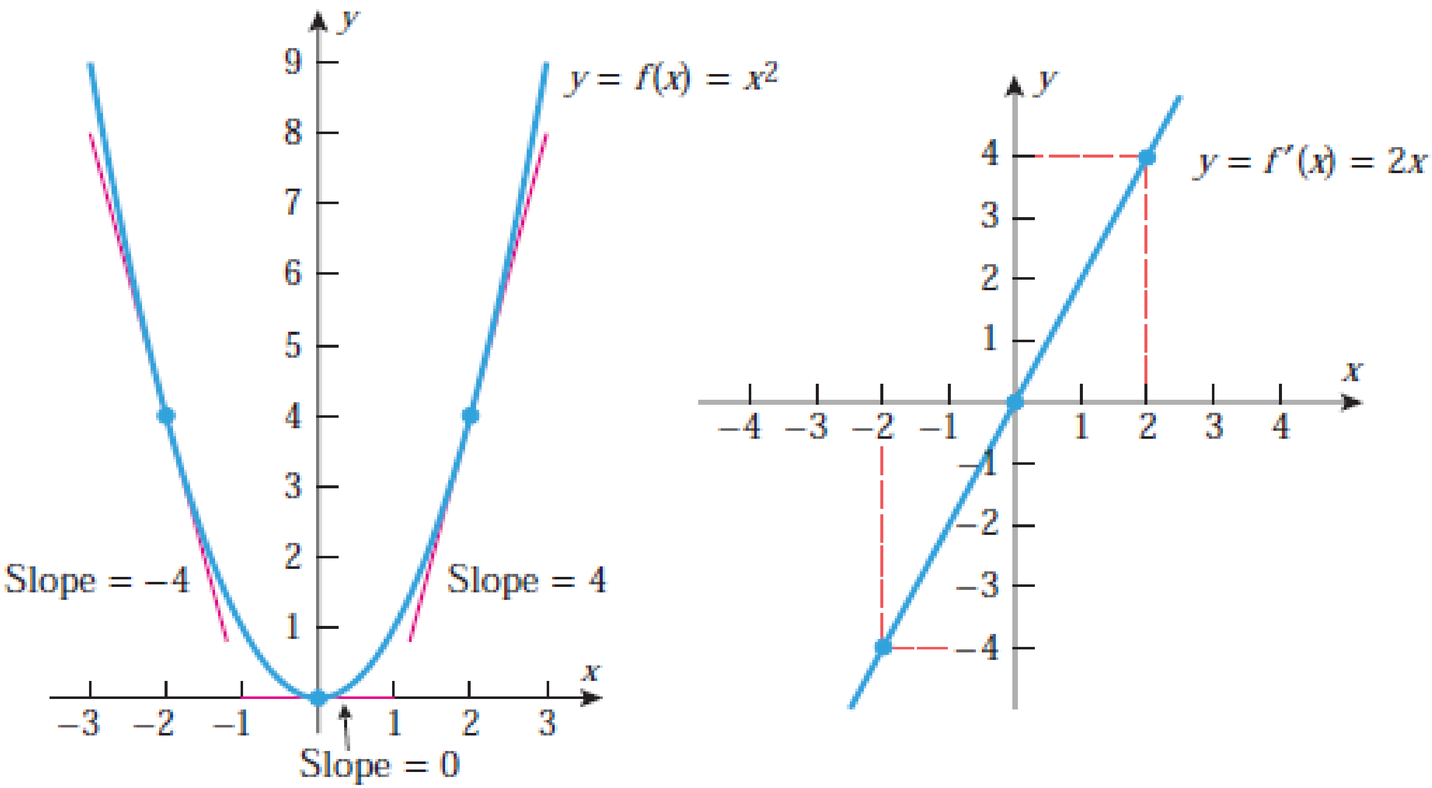
You can think of f' as a "slope-producing function" in the sense that the value of $f'(x)$ at $x = x_0$ is the slope of the tangent line to the graph of f at $x = x_0$. This aspect of the derivative is illustrated in Figure 2.2.2, which shows the graphs of $f(x) = x^2$ and its derivative $f'(x) = 2x$ (obtained in Example 1). The figure illustrates that the values of $f'(x) = 2x$ at $x = -2, 0,$ and 2 correspond to the slopes of the tangent lines to the graph of $f(x) = x^2$ at those values of x.



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In general, if $f'(x)$ is defined at $x = x_0$, then the point-slope form of the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$ may be found using the following steps.

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{3}$$

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▶ Example 2

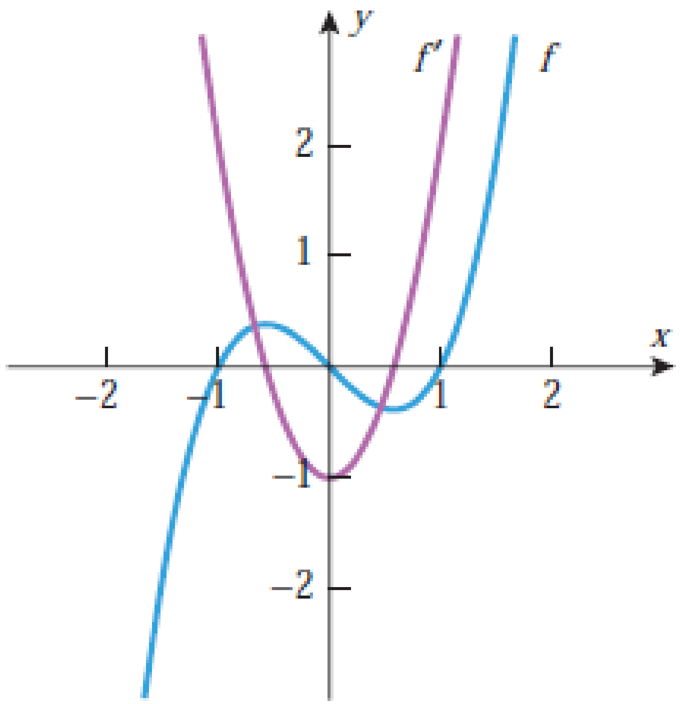
(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

(b) Graph f and f' together, and discuss the relationship between the two graphs.

Solution (a).

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h \to 0} [3x^2 + 3xh + h^2 - 1] = 3x^2 - 1
\end{aligned}
$$

Solution (b). Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that $f'(x)$ is positive where the tangent line has positive slope, is negative where the tangent line has negative slope, and is zero where the tangent line is horizontal. We leave it for you to verify that this is consistent with the graphs of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$ shown in Figure 2.2.3. ◀



▲ Figure 2.2.3

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▶ Example 4

(a) Find the derivative with respect to x of $f(x) = \sqrt{x}$.

(b) Find the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$.

Solution (a). Recall from Example 4 of Section 2.1 that the slope of the tangent line to $y = \sqrt{x}$ at $x = x_0$ is given by $m_{\tan} = 1/(2\sqrt{x_0})$. Thus, $f'(x) = 1/(2\sqrt{x})$.

Solution (b). The slope of the tangent line at $x = 9$ is $f'(9)$. From part (a), this slope is $f'(9) = 1/(2\sqrt{9}) = \frac{1}{6}$.

COMPUTING INSTANTANEOUS VELOCITY

It follows from Formula (5) of Section 2.1 (with t replacing t_0) that if $s = f(t)$ is the position function of a particle in rectilinear motion, then the instantaneous velocity at an arbitrary time t is given by

$$v_{\text{inst}} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Since the right side of this equation is the derivative of the function f (with t rather than x as the independent variable), it follows that if $f(t)$ is the position function of a particle in rectilinear motion, then the function

$$v(t) = f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \qquad (4)$$

represents the instantaneous velocity of the particle at time t. Accordingly, we call (4) the *instantaneous velocity function* or, more simply, the *velocity function* of the particle.

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▶ Example 5 Recall the particle from Example 5 of Section 2.1 with position function $s = f(t) = 1 + 5t - 2t^2$. Here $f(t)$ is measured in meters and t is measured in seconds. Find the velocity function of the particle.

Solution. It follows from (4) that the velocity function is

$$\begin{aligned}
v(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{[1 + 5(t+h) - 2(t+h)^2] - [1 + 5t - 2t^2]}{h} \\
&= \lim_{h\to 0} \frac{-2[t^2 + 2th + h^2 - t^2] + 5h}{h} = \lim_{h\to 0} \frac{-4th - 2h^2 + 5h}{h} \\
&= \lim_{h\to 0} (-4t - 2h + 5) = 5 - 4t
\end{aligned}$$

where the units of velocity are meters per second. ◀

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DIFFERENTIABILITY

It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined. To account for this possibility we make the following definition.

2.2.2 DEFINITION A function f is said to be *differentiable at* x_0 if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{5}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is *differentiable on* (a, b), and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is *differentiable everywhere*.

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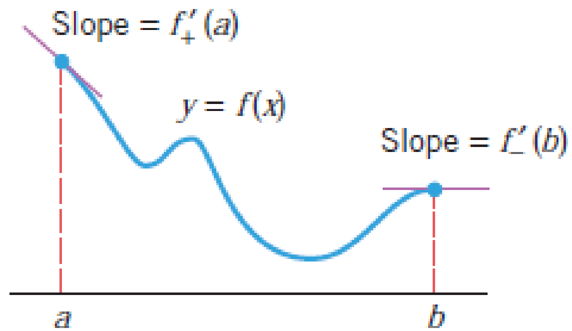
DERIVATIVES AT THE ENDPOINTS OF AN INTERVAL

If a function f is defined on a closed interval $[a, b]$ but not outside that interval, then f' is not defined at the endpoints of the interval because derivatives are two-sided limits. To deal with this we define ***left-hand derivatives*** and ***right-hand derivatives*** by

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

respectively. These are called ***one-sided derivatives***. Geometrically, $f'_-(x)$ is the limit of the slopes of the secant lines as x is approached from the left and $f'_+(x)$ is the limit of the slopes of the secant lines as x is approached from the right. For a closed interval $[a, b]$, we will understand the derivative at the left endpoint to be $f'_+(a)$ and at the right endpoint to be $f'_-(b)$ (Figure 2.2.13).



▲ **Figure 2.2.13**

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In general, we will say that f is *differentiable* on an interval of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, or $(a, b]$ if it is differentiable at all points inside the interval and the appropriate one-sided derivative exists at each included endpoint.

It can be proved that a function f is continuous from the left at those points where the left-hand derivative exists and is continuous from the right at those points where the right-hand derivative exists.

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DERIVATIVE NOTATIONS

- The process of finding a derivative is called ***differentiation***. You can think of differentiation as an *operation* on functions that associates a function f' with a function f. When the independent variable is x, the differentiation operation is also commonly denoted by

$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)]$$

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- In the case where there is a dependent variable *y = f(x)*, the derivative is also commonly denoted by

$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}$$

- With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x=x_0}, \quad f'(x_0) = D_x[f(x)]\big|_{x=x_0}, \quad f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\bigg|_{x=x_0}$$

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- If a variable *w* changes from some initial value w_0 to some final value w_1, then the final value minus the initial value is called an ***increment*** in *w* and is denoted by

$$\Delta w = w_1 - w_0 \tag{8}$$

Increments can be positive or negative, depending on whether the final value is larger or smaller than the initial value. The increment symbol in (8) should not be interpreted as a product; rather, Δw should be regarded as a single symbol representing the change in the value of w.

It is common to regard the variable h in the derivative formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

as an increment Δx in x and write (9) as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{10}$$

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Moreover, if $y = f(x)$, then the numerator in (10) can be regarded as the increment

$$\Delta y = f(x + \Delta x) - f(x) \tag{11}$$

in which case

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{12}$$

The geometric interpretations of Δx and Δy are shown in Figure 2.2.14.

Sometimes it is desirable to express derivatives in a form that does not use increments at all. For example, if we let $w = x + h$ in Formula (9), then $w \to x$ as $h \to 0$, so we can rewrite that formula as

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x} \tag{13}$$

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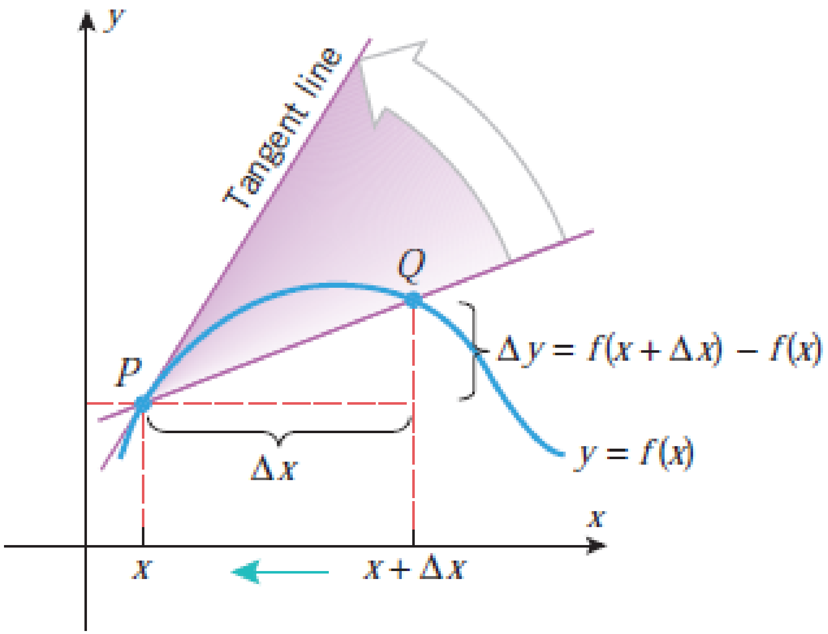
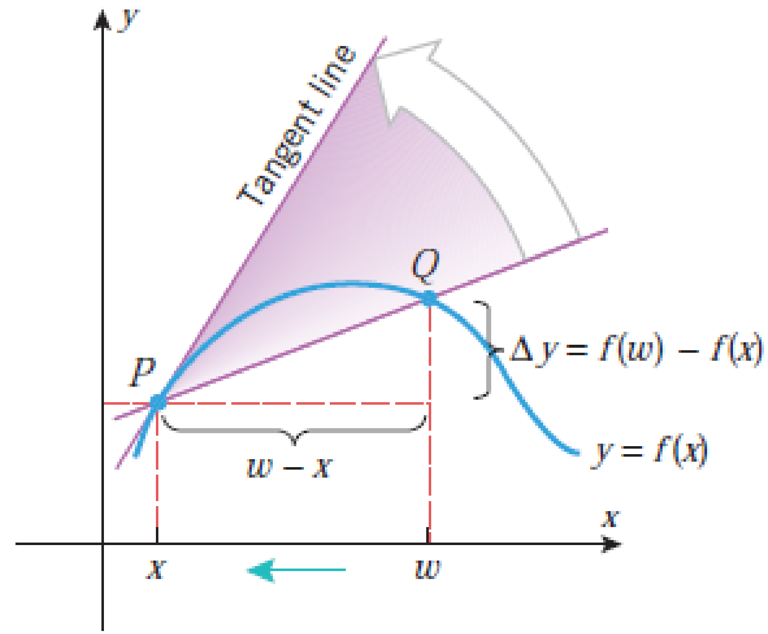
(Compare Figures 2.2.14 and 2.2.15.)



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

▲ **Figure 2.2.14**



$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

▲ **Figure 2.2.15**

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When letters other than x and y are used for the independent and dependent variables, the derivative notations must be adjusted accordingly. Thus, for example, if $s = f(t)$ is the position function for a particle in rectilinear motion, then the velocity function $v(t)$ in (4) can be expressed as

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{14}$$

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Example

Find dy/dx, given $y = x^3 - x^2 - 4$. Find also the value of dy/dx when (a) $x = 4$, (b) $x = 0$, (c) $x = -1$.

$$y + \Delta y = (x + \Delta x)^3 - (x + \Delta x)^2 - 4$$

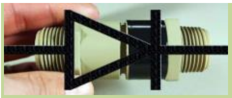
$$= x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^2 - 2x(\Delta x) - (\Delta x)^2 - 4 \quad -y$$

$$= [x^3 + 3x^2(\Delta x) + 3x(\Delta x) + (\Delta x)^3 - x^2 - 2x(\Delta x) - (\Delta x)^2 - 4)\ ] - [(x^3 - x^2 - 4)]$$

$$\Delta y = (3x^2 - 2x)\Delta x + (3x - 1)(\Delta x)^2 + (\Delta x)^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} [3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2] = 3x^2 - 2x$$



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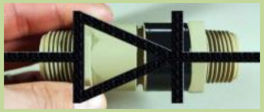
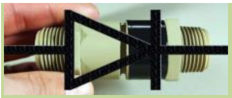


(a) $$\left.\frac{dy}{dx}\right|_{x=4} = 3(4)^2 - 2(4) = 40;$$

(b) $$\left.\frac{dy}{dx}\right|_{x=0} = 3(0)^2 - 2(0) = 0;$$

(c) $$\left.\frac{dy}{dx}\right|_{x=-1} = 3(-1)^2 - 2(-1) = 5$$



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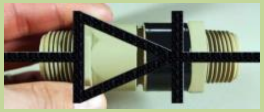

Find the derivative of $y = f(x) = x^2 + 3x + 5$.

$$\Delta y = f(x+\Delta x) - f(x) = [(x+\Delta x)^2 + 3(x+\Delta x) + 5)] - [x^2 + 3x + 5]$$

$$= [x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 5] - [x^2 + 3x + 5] = 2x\Delta x + (\Delta x)^2 + 3\Delta x$$

$$= (2x + \Delta x + 3)\Delta x$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 3$$

$$\text{So, } \frac{dy}{dx} = \lim_{\Delta x \to 0} (2x + \Delta x + 3) = 2x + 3.$$

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Find the derivative of each of the following: using increment formula

(a) $y = 4x - 3$

(b) $y = 4 - 3x$

(c) $y = x^2 + 2x - 3$

(d) Find the derivative of $f(x) = \dfrac{2x - 3}{3x + 4}$.

(e) Find the derivative of $y = f(x) = \dfrac{1}{x - 2}$ at $x = 1$ and $x = 3$.

DERIVATIVES

Find the derivative of each of the following: using increment formula

(a) $y = 4x - 3$

(b) $y = 4 - 3x$

(c) $y = x^2 + 2x - 3$

(d) Find the derivative of $f(x) = \dfrac{2x-3}{3x+4}$.

(e) Find the derivative of $y = f(x) = \dfrac{1}{x-2}$ at $x = 1$ and $x = 3$.

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Using this formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example . If $f(x) = 3x^2 - 12x + 8$. find

(a) $f'(x)$

(b) $f'(4)$

(c) $f'(-2)$

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Solution

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 12(x+h) + 8] - (3x^2 - 12x + 8)}{h} \\
&= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 - 12x - 12h + 8) - (3x^2 - 12x + 8)}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 12h}{h} \\
&= \lim_{h \to 0} (6x + 3h - 12) \\
&= 6x - 12.
\end{aligned}
$$

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Solution

b. $f'(4)$

Substituting for x in $f'(x) = 6x - 12$, we obtain

$$f'(4) = 6(4) - 12 = 12,$$

$$f'(-2) = 6(-2) - 12 = -24.$$

c. $f'(x) = 6x - 12$

$$f'(-2) = 6(-2) - 12$$
$$= -24$$

DERIVATIVES

INTRODUCTION TO TECHNIQUES OF DIFFERENTIATION

DERIVATIVE OF A CONSTANT

2.3.1 THEOREM *The derivative of a constant function is 0; that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0 \qquad (1)$$

Example 1

$$\frac{d}{dx}[1] = 0, \quad \frac{d}{dx}[-3] = 0, \quad \frac{d}{dx}[\pi] = 0, \quad \frac{d}{dx}\left[-\sqrt{2}\right] = 0$$

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2.3.2 THEOREM (*The Power Rule*) *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

PROOF Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $(x+h)^n$, we obtain

$$\frac{d}{dx}[x^n] = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1} + 0 + \cdots + 0 + 0$$

$$= nx^{n-1} \quad \blacksquare$$

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▶ Example 2

$$\frac{d}{dx}[x^4] = 4x^3, \quad \frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dt}[t^{12}] = 12t^{11}$$ ◀

2.3.3 THEOREM (*Extended Power Rule*) *If r is any real number, then*

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{7}$$

In words, *to differentiate a power function, decrease the constant exponent by one and multiply the resulting power function by the original exponent.*

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2.3.3 THEOREM (*Extended Power Rule*) *If r is any real number, then*

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{7}$$

In words, *to differentiate a power function, decrease the constant exponent by one and multiply the resulting power function by the original exponent.*

► **Example 3**

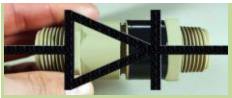
$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dw}\left[\frac{1}{w^{100}}\right] = \frac{d}{dw}[w^{-100}] = -100w^{-101} = -\frac{100}{w^{101}}$$

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$ ◄



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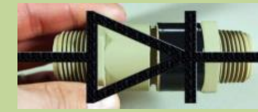

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

2.3.4 THEOREM (*Constant Multiple Rule*) *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \tag{8}$$

PROOF

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= c\frac{d}{dx}[f(x)] \quad \blacksquare\end{aligned}$$

A constant factor can be moved through a limit sign.

Formula (8) can also be expressed in function notation as

$$(cf)' = cf'$$

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In words, *a constant factor can be moved through a derivative sign.*

▶ Example 4

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{\pi}{x}\right] = \pi\frac{d}{dx}[x^{-1}] = \pi(-x^{-2}) = -\frac{\pi}{x^2}$$

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DERIVATIVES OF SUMS AND DIFFERENCES

2.3.5 THEOREM (*Sum and Difference Rules*) *If f and g are differentiable at x, then so are $f+g$ and $f-g$ and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \tag{9}$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \tag{10}$$

PROOF Formula (9) can be proved as follows:

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]
\end{aligned}$$

The limit of a sum is the sum of the limits.

DERIVATIVES

▶ Example 5

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}] = 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\frac{d}{dx}\left[\frac{\sqrt{x} - 2x}{\sqrt{x}}\right] = \frac{d}{dx}[1 - 2\sqrt{x}\,]$$

$$= \frac{d}{dx}[1] - \frac{d}{dx}[2\sqrt{x}\,] = 0 - 2\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x}}$$

See Formula (6). ◀

Although Formulas (9) and (10) are stated for sums and differences of two functions, they can be extended to any finite number of functions. For example, by grouping and applying Formula (9) twice we obtain

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

As illustrated in the following example, the constant multiple rule together with the extended versions of the sum and difference rules can be used to differentiate any polynomial.

DERIVATIVES

► **Example 6** Find dy/dx if $y = 3x^8 - 2x^5 + 6x + 1$.

Solution.

$$\frac{dy}{dx} = \frac{d}{dx}[3x^8 - 2x^5 + 6x + 1]$$

$$= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]$$

$$= 24x^7 - 10x^4 + 6$$ ◄

THE PRODUCT AND QUOTIENT RULES

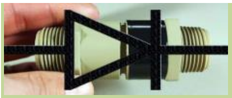
DERIVATIVE OF A PRODUCT

2.4.1 THEOREM (*The Product Rule*) *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

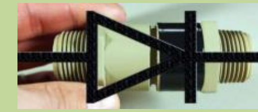
$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{1}$$

Formula (1) can also be expressed as

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$



DERIVATIVES



PROOF Whereas the proofs of the derivative rules in the last section were straightforward applications of the derivative definition, a key step in this proof involves adding and subtracting the quantity $f(x+h)g(x)$ to the numerator in the derivative definition. This yields

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[f(x+h)\cdot \frac{g(x+h)-g(x)}{h} + g(x)\cdot \frac{f(x+h)-f(x)}{h} \right]$$

$$= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \left[\lim_{h\to 0} f(x+h) \right] \frac{d}{dx}[g(x)] + \left[\lim_{h\to 0} g(x) \right] \frac{d}{dx}[f(x)]$$

$$= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

DERIVATIVES

▶ **Example 1** Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution. There are two methods that can be used to find dy/dx. We can either use t product rule or we can multiply out the factors in y and then differentiate. We will gi both methods.

Method 1. (*Using the Product Rule*)

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)]$$

$$= (4x^2 - 1)\frac{d}{dx}[7x^3 + x] + (7x^3 + x)\frac{d}{dx}[4x^2 - 1]$$

$$= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x) = 140x^4 - 9x^2 - 1$$

Method 2. (*Multiplying First*)

$$y = (4x^2 - 1)(7x^3 + x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

which agrees with the result obtained using the product rule. ◀

DERIVATIVES

Example 2 Find ds/dt if $s = (1+t)\sqrt{t}$.

Solution. Applying the product rule yields

$$\begin{aligned}
\frac{ds}{dt} &= \frac{d}{dt}[(1+t)\sqrt{t}] \\
&= (1+t)\frac{d}{dt}[\sqrt{t}] + \sqrt{t}\frac{d}{dt}[1+t] \\
&= \frac{1+t}{2\sqrt{t}} + \sqrt{t} = \frac{1+3t}{2\sqrt{t}} \quad \blacktriangleleft
\end{aligned}$$

DERIVATIVES

DERIVATIVE OF A QUOTIENT

- Just as the derivative of a product is not generally the product of the derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship is given by the following theorem.

2.4.2 THEOREM (*The Quotient Rule*) *If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \tag{2}$$

Formula (2) can also be expressed as

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

DERIVATIVES

PROOF

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Adding and subtracting $f(x)\cdot g(x)$ in the numerator yields

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)}$$

$$= \lim_{h\to 0}\frac{\left[g(x)\cdot\dfrac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot\dfrac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)}$$

$$= \frac{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0} f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)}$$

$$= \frac{\left[\lim\limits_{h\to 0} g(x)\right]\cdot\dfrac{d}{dx}[f(x)] - \left[\lim\limits_{h\to 0} f(x)\right]\cdot\dfrac{d}{dx}[g(x)]}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)}$$

$$= \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

DERIVATIVES

Example 3 Find $y'(x)$ for $y = \dfrac{x^3 + 2x^2 - 1}{x + 5}$.

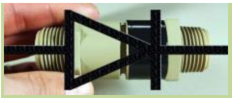
Solution. Applying the quotient rule yields

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3 + 2x^2 - 1}{x + 5}\right] = \frac{(x+5)\dfrac{d}{dx}[x^3 + 2x^2 - 1] - (x^3 + 2x^2 - 1)\dfrac{d}{dx}[x+5]}{(x+5)^2}$$

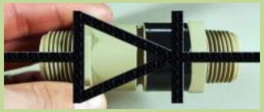
$$= \frac{(x+5)(3x^2 + 4x) - (x^3 + 2x^2 - 1)(1)}{(x+5)^2}$$

$$= \frac{(3x^3 + 19x^2 + 20x) - (x^3 + 2x^2 - 1)}{(x+5)^2}$$

$$= \frac{2x^3 + 17x^2 + 20x + 1}{(x+5)^2}$$



DERIVATIVES



Example 4 Let $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

FIND f'(x)

Solution. Applying the quotient rule yields

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2-1}{x^4+1}\right] = \frac{(x^4+1)\dfrac{d}{dx}[x^2-1] - (x^2-1)\dfrac{d}{dx}[x^4+1]}{(x^4+1)^2}$$

$$= \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2}$$

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2}$$

$$= -\frac{2x(x^4 - 2x^2 - 1)}{(x^4+1)^2}$$

DERIVATIVES

SUMMARY OF DIFFERENTIATION RULES

RULES FOR DIFFERENTIATION

$$\frac{d}{dx}[c] = 0$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

Assignment

Find the derivative of the following functions

1. $y = 2x^2$
2. $y = 2x^{-2}$
3. $y = \dfrac{100}{x^5}$
4. $y = x^2 + 2x$
5. $y = x^4 + x^3 + x^2 + x + 1$
6. $y = \dfrac{3}{x^3} + x^{-4}$
7. $y = 2x^{-6} + x^{-1}$
8. $y = \pi x^7 - 2x^5 - 5x^{-2}$
9. $y = 3x^4 - 2x^3 - 5x^2 + \pi x + \pi^2$
10. $y = (2x + 1)^2$