CALCULUS 1

DERIVATIVES

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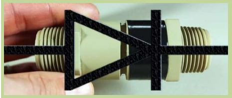
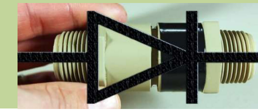


OBJECTIVES



- Discuss the derivative function

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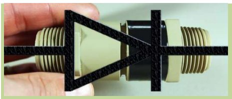
THE DERIVATIVE FUNCTION

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (2)$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

The term “derivative” is used because the function *f* is *derived* from the function *f* by a limiting process.



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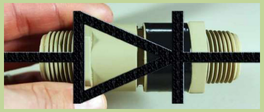

Example 1

Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

Solution. It follows from (2) that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} (2x + h) = 2x$$

Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$. Since $y = 4$ if $x = 2$, the point-slope form of the tangent line is

$$y - 4 = 4(x - 2)$$

which we can rewrite in slope-intercept form as $y = 4x - 4$ (Figure 2.2.1). ◀

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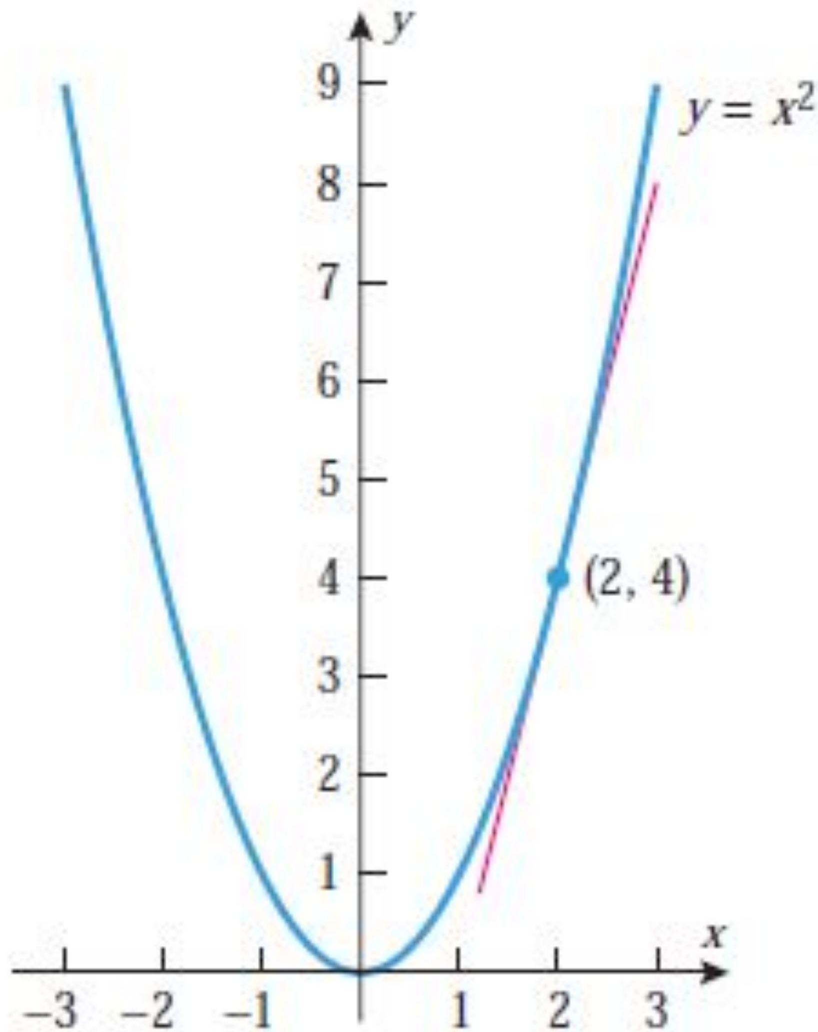

▲ **Figure 2.2.1**

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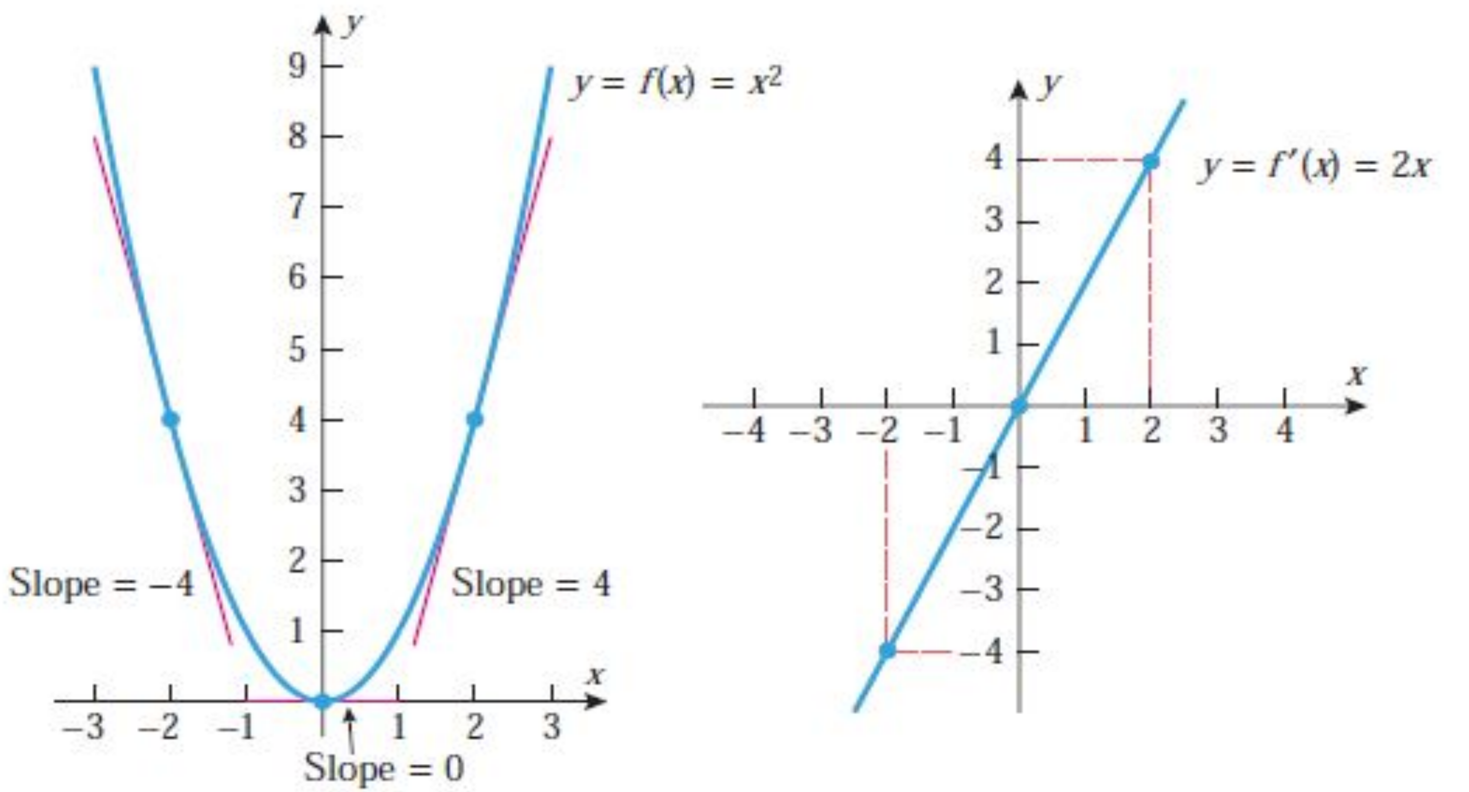
You can think of f' as a "slope-producing function" in the sense that the value of $f'(x)$ at $x = x_0$ is the slope of the tangent line to the graph of f at $x = x_0$. This aspect of the derivative is illustrated in Figure 2.2.2, which shows the graphs of $f(x) = x^2$ and its derivative $f'(x) = 2x$ (obtained in Example 1). The figure illustrates that the values of $f'(x) = 2x$ at $x = -2, 0$, and 2 correspond to the slopes of the tangent lines to the graph of $f(x) = x^2$ at those values of x.



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In general, if $f'(x)$ is defined at $x = x_0$, then the point-slope form of the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$ may be found using the following steps.

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \qquad (3)$$

▸ Example 2

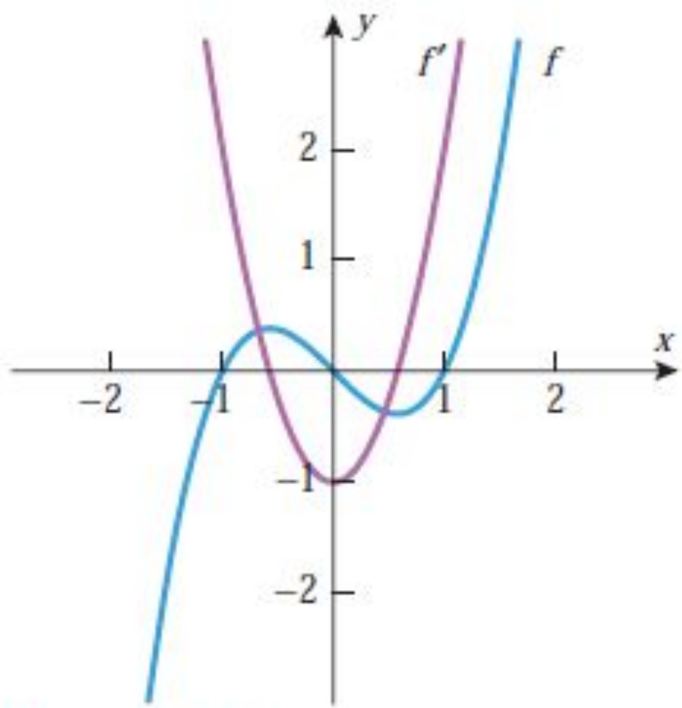
(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

(b) Graph f and f' together, and discuss the relationship between the two graphs.

Solution (a).

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h \to 0} [3x^2 + 3xh + h^2 - 1] = 3x^2 - 1
\end{aligned}
$$

Solution (b). Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that $f'(x)$ is positive where the tangent line has positive slope, is negative where the tangent line has negative slope, and is zero where the tangent line is horizontal. We leave it for you to verify that this is consistent with the graphs of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$ shown in Figure 2.2.3. ◄



▲ **Figure 2.2.3**

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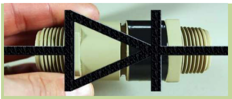
▶ Example 4

(a) Find the derivative with respect to x of $f(x) = \sqrt{x}$.

(b) Find the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$.

Solution (a). Recall from Example 4 of Section 2.1 that the slope of the tangent line to $y = \sqrt{x}$ at $x = x_0$ is given by $m_{\tan} = 1/(2\sqrt{x_0})$. Thus, $f'(x) = 1/(2\sqrt{x})$.

Solution (b). The slope of the tangent line at $x = 9$ is $f'(9)$. From part (a), this slope is $f'(9) = 1/(2\sqrt{9}) = \frac{1}{6}$.



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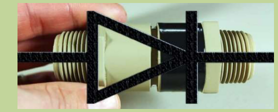

COMPUTING INSTANTANEOUS VELOCITY

It follows from Formula (5) of Section 2.1 (with t replacing t_0) that if $s = f(t)$ is the position function of a particle in rectilinear motion, then the instantaneous velocity at an arbitrary time t is given by

$$v_{\text{inst}} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Since the right side of this equation is the derivative of the function f (with t rather than x as the independent variable), it follows that if $f(t)$ is the position function of a particle in rectilinear motion, then the function

$$v(t) = f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \tag{4}$$

represents the instantaneous velocity of the particle at time t. Accordingly, we call (4) the *instantaneous velocity function* or, more simply, the *velocity function* of the particle.

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▸ **Example 5** Recall the particle from Example 5 of Section 2.1 with position function $s = f(t) = 1 + 5t - 2t^2$. Here $f(t)$ is measured in meters and t is measured in seconds. Find the velocity function of the particle.

Solution. It follows from (4) that the velocity function is

$$v(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{[1 + 5(t+h) - 2(t+h)^2] - [1 + 5t - 2t^2]}{h}$$

$$= \lim_{h\to 0} \frac{-2[t^2 + 2th + h^2 - t^2] + 5h}{h} = \lim_{h\to 0} \frac{-4th - 2h^2 + 5h}{h}$$

$$= \lim_{h\to 0} (-4t - 2h + 5) = 5 - 4t$$

where the units of velocity are meters per second. ◂

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DIFFERENTIABILITY

It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined. To account for this possibility we make the following definition.

2.2.2 DEFINITION A function f is said to be ***differentiable at*** x_0 if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{5}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on*** (a, b), and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is ***differentiable everywhere***.

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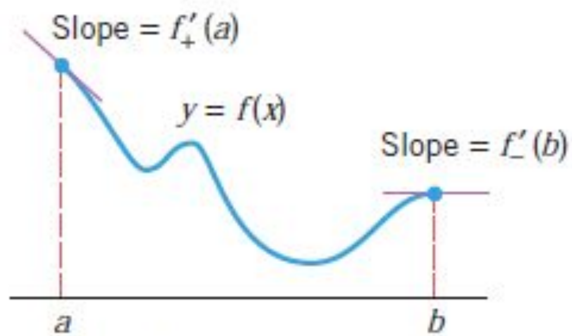
DERIVATIVES AT THE ENDPOINTS OF AN INTERVAL

If a function f is defined on a closed interval $[a, b]$ but not outside that interval, then f' is not defined at the endpoints of the interval because derivatives are two-sided limits. To deal with this we define ***left-hand derivatives*** and ***right-hand derivatives*** by

$$f'_-(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \quad \text{and} \quad f'_+(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

respectively. These are called ***one-sided derivatives***. Geometrically, $f'_-(x)$ is the limit of the slopes of the secant lines as x is approached from the left and $f'_+(x)$ is the limit of the slopes of the secant lines as x is approached from the right. For a closed interval $[a, b]$, we will understand the derivative at the left endpoint to be $f'_+(a)$ and at the right endpoint to be $f'_-(b)$ (Figure 2.2.13).



▲ **Figure 2.2.13**

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In general, we will say that f is *differentiable* on an interval of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, or $(a, b]$ if it is differentiable at all points inside the interval and the appropriate one-sided derivative exists at each included endpoint.

It can be proved that a function f is continuous from the left at those points where the left-hand derivative exists and is continuous from the right at those points where the right-hand derivative exists.