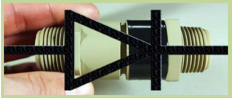
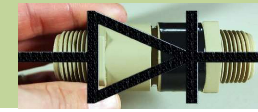
LESSON 4

CONTINUITY

PRESENTED BY ENGR. JOHN R. REJANO,ECE



OBJECTIVES



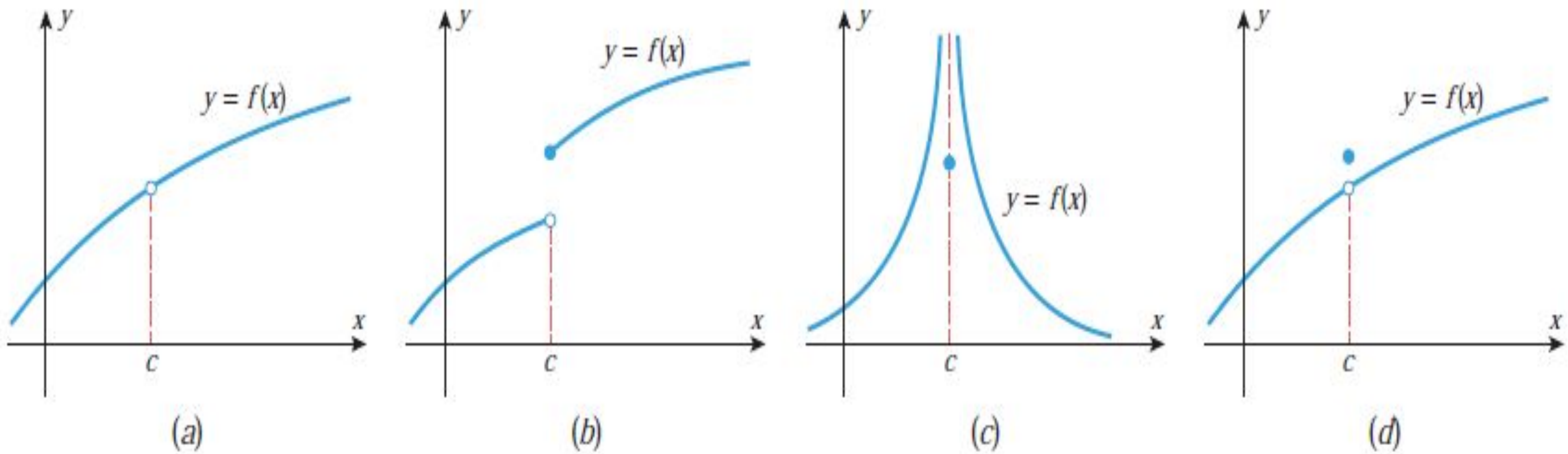
- ◆ Discuss the continuity
- ◆ Discuss Intermediate value theorem

Discuss removable and jump discontinuity

CONTINUITY

- Intuitively, the graph of a function can be described as a “continuous curve” if it has no breaks or holes. To make this idea more precise we need to understand what properties of a function can cause breaks or holes.

Referring to Figure 1.5.1, we see that the graph of a function has a break or hole if any of the following conditions occur:



▲ Figure 1.5.1

CONTINUITY

This suggests the following definition.

1.5.1 DEFINITION A function f is said to be ***continuous at*** $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

If one or more of the conditions of this definition fails to hold, then we will say that *f* has a ***discontinuity at x = c***.

CONTINUITY

Example 1 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

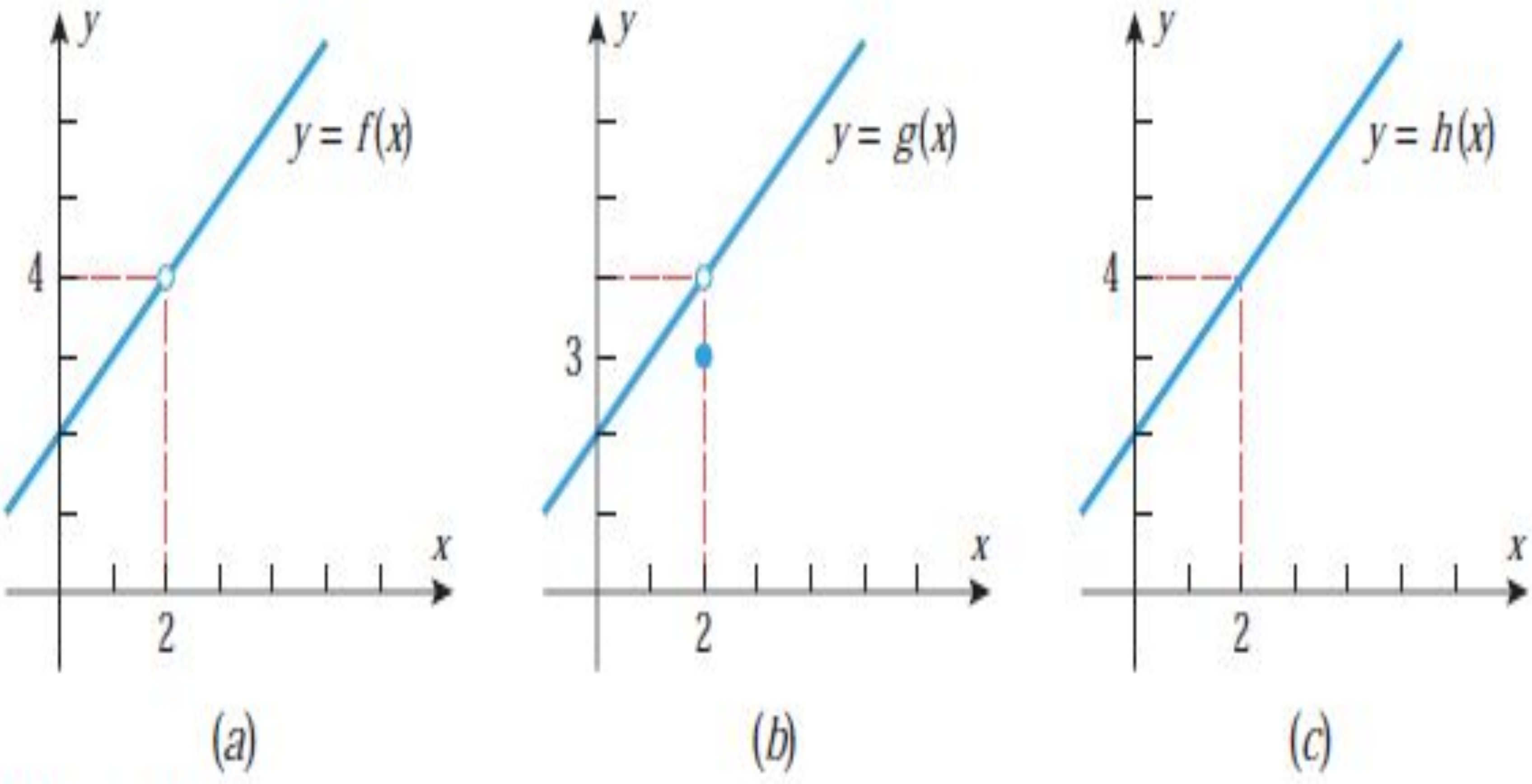
CONTINUITY

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 1.5.2*a*). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches 2; hence, g is also not continuous at $x = 2$ (Figure 1.5.2*b*). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches 2; hence, h is continuous at $x = 2$ (Figure 1.5.2*c*). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀

CONTINUITY



▲ Figure 1.5.2

CONTINUITY

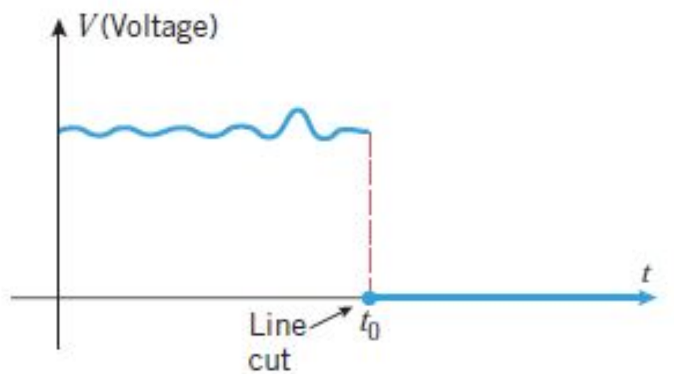
CONTINUITY IN APPLICATIONS

- In applications, discontinuities often signal the occurrence of important physical events. For example, Figure 1.5.3*a* is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time $t = t_0$

(the voltage drops to zero when the line is cut).

Chris Hondros/Getty Images

A poor connection in a transmission cable can cause a discontinuity in the electrical signal it carries.



(*a*)

▲ **Figure 1.5.3**

CONTINUITY

Chris Hondros/Getty Images

A poor connection in a transmission cable can cause a discontinuity in the electrical signal it carries.

Figure 1.5.3*b* shows the graph of inventory versus time for a company that restocks its warehouse to y_1 units when the inventory falls to y_0 units. The discontinuities occur at those times when restocking occurs

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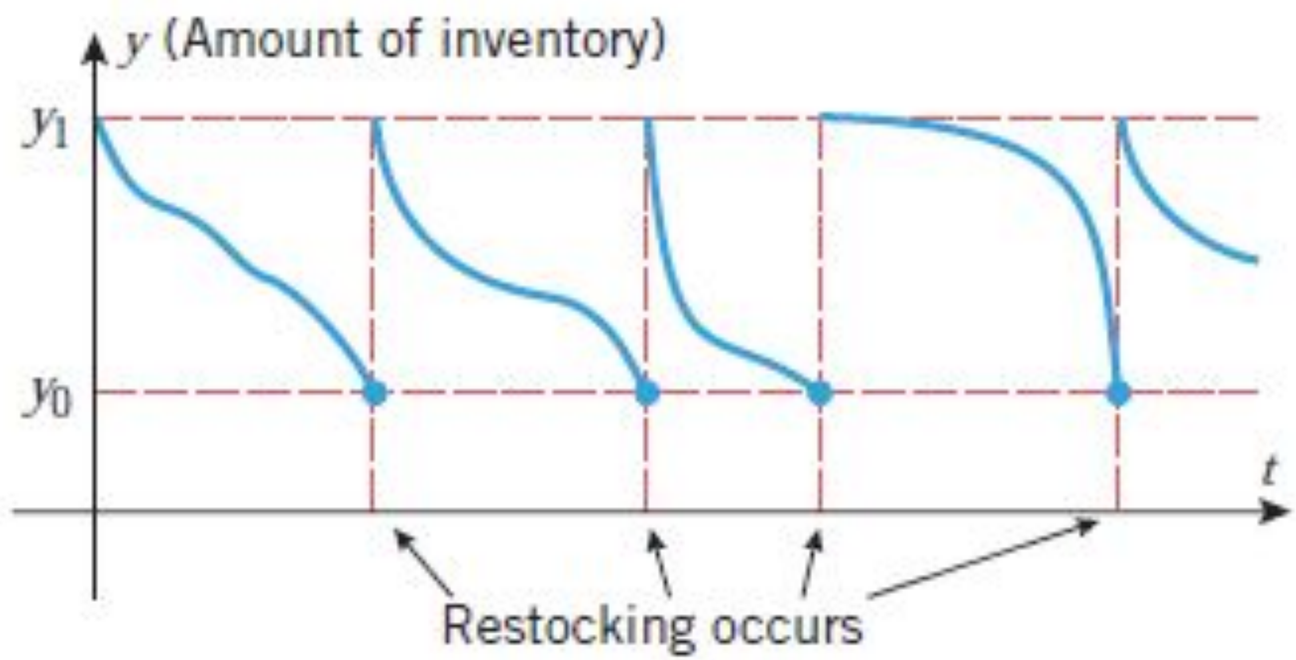


(*b*)

CONTINUITY

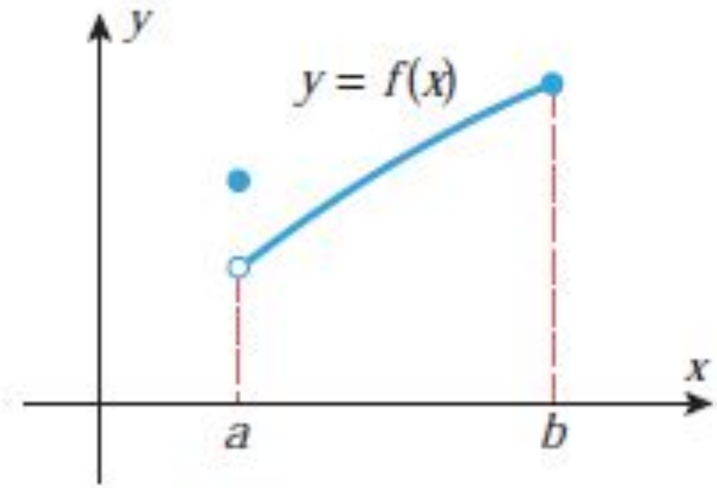
CONTINUITY ON AN INTERVAL

If a function *f* is continuous at each number in an open interval *(a, b)*, then we say that *f* is ***continuous on*** **(*a, b*)**. This definition applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$.

In the case where *f* is continuous on *(−∞,+∞)*, we will say that *f* is ***continuous everywhere***.
To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. For example, the function graphed in Figure 1.5.4 is continuous at the right endpoint of the interval [a, b] because

$$\lim_{x \to b^-} f(x) = f(b)$$

CONTINUITY



▲ **Figure 1.5.4**

but it is not continuous at the left endpoint because

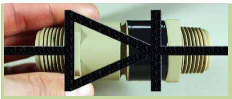
$$\lim_{x \to a^+} f(x) \neq f(a)$$

In general, we will say a function f is *continuous from the left* at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

and is *continuous from the right* at c if

$$\lim_{x \to c^+} f(x) = f(c)$$



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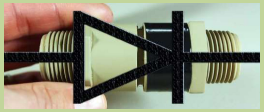


Using this terminology we define continuity on a closed interval as follows.

1.5.2 DEFINITION A function f is said to be *continuous on a closed interval* $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

CONTINUITY

▶ **Example 2** What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then it follows from Theorem 1.2.2(*e*) that

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

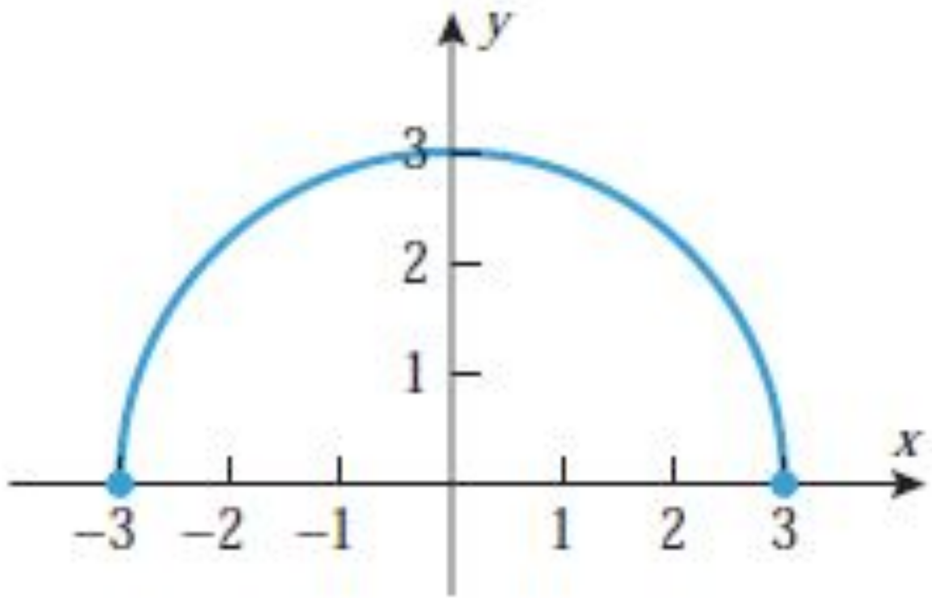
which proves f is continuous at each point in the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$

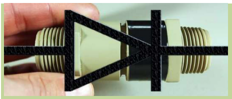
Thus, f is continuous on the closed interval $[-3, 3]$ (Figure 1.5.5). ◀

CONTINUITY



$$f(x) = \sqrt{9 - x^2}$$

Figure 1.5.5



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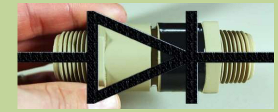


SOME PROPERTIES OF CONTINUOUS FUNCTIONS

The following theorem, which is a consequence of Theorem 1.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting,multiplying, and dividing continuous functions.

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(a) $f + g$ *is continuous at* c.

(b) $f - g$ *is continuous at* c.

(c) fg *is continuous at* c.

(d) f/g *is continuous at* c *if* $g(c) \neq 0$ *and has a discontinuity at* c *if* $g(c) = 0$.

CONTINUITY

PROOF First, consider the case where $g(c) = 0$. In this case $f(c)/g(c)$ is undefined, so the function f/g has a discontinuity at c.

Next, consider the case where $g(c) \neq 0$. To prove that f/g is continuous at c, we must show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{1}$$

Since f and g are continuous at c,

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)$$

Thus, by Theorem 1.2.2(d)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (1). ■

CONTINUITY

CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS

1.5.4 THEOREM

(a) *A polynomial is continuous everywhere.*

(b) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

CONTINUITY

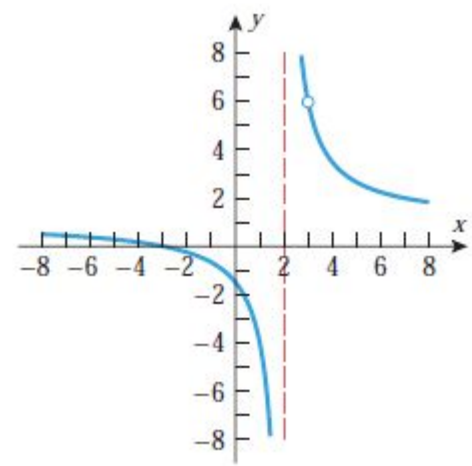
Example 3 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

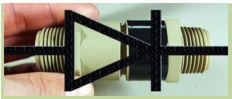
$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$ (Figure 1.5.6). ◀

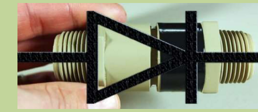


$$y = \frac{x^2 - 9}{x^2 - 5x + 6}$$

▲ **Figure 1.5.6**



exercises



1. What three conditions are satisfied if *f* is continuous at $x = c$?

2. For what values of *x*, if any, is the function

$$f(x) = \frac{x^2 - 16}{x^2 - 5x + 4}$$ discontinuous?

3. Find values of *x*, if any, at which *f* is not continuous.

a. $f(x) = 5x^4 - 3x + 7$

b. $f(x) = \dfrac{x+2}{x^2+4}$

c. $f(x) = \dfrac{3}{x} + \dfrac{x-1}{x^2-1}$

CONTINUITY

REMOVABLE DISCONTINUITY

A function f(x) has a removable discontinuity at c if and only if

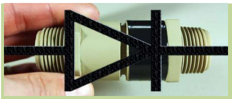
i. $\lim_{x \to c} f(x)$ exists

ii. f(c) is defined

iii. $\lim\limits_{x \to c} f(x) \neq c$

Steps in solving removable discontinuity

1. Find all its discontinuity
2. Redefined the function if the discontinuity is removable



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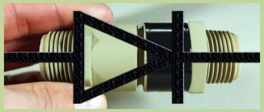


Example1: Solve for removable discontinuity

$$f(x) = \frac{x^2 - 4}{x - 2}$$

1. Find all its discontinuity

The function is undefined at x = 2

2. Write the function in reduced form (use factoring method to eliminate the term that makes it undefined)

$$f(x) = \frac{(x+2)(x-2)}{x-2}$$

$$f(x) = x + 2$$

$$\lim_{x \to 2^-} (x+2) = 4 \qquad \lim_{x \to 2^+} (x+2) = 4$$

CONTINUITY

3 Redefined the function if the discontinuity is removable. To redefined the function put in piecewise function form.

$$f(x) = \begin{cases} \dfrac{x^2 - 4^2}{x - 2}, & if\ x \neq 0 \\ 4, & x = 2 \end{cases}$$

CONTINUITY

EXAMPLE 8.3: Let f be the function such that $f(x) = \frac{|x|}{x}$ for all $x \neq 0$. The graph of f is shown in Fig. 8-3. f is discontinuous at 0 because $f(0)$ is not defined. Moreover,

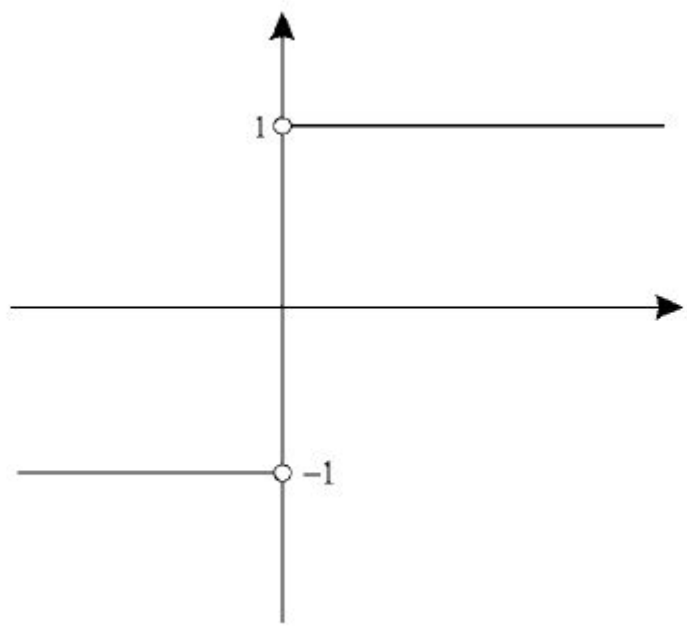


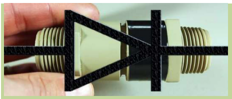
Fig. 8-3

CONTINUITY

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

Thus, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. Hence, the discontinuity of f at 0 is not removable.

The kind of discontinuity shown in Example 8.3 is called a *jump discontinuity*. In general, a function f has a jump discontinuity at x_0 if $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ both exist and $\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x)$. Such a discontinuity is not removable.



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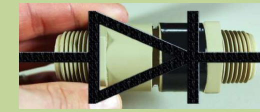

1. CALCULUS by H. Anton, et al 10th edition
2. Schaum's outline series CALCULUS 6th edition by Ayres/ Mendelson