LESSON 4

LIMITS

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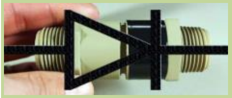
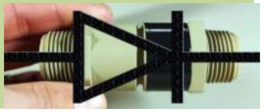
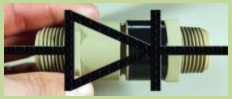


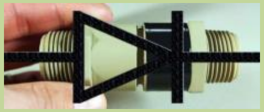
OBJECTIVES



- ◆ Discus limits of piecewise and trigonometric functions
- ◆ Discus limits exponential and logarithmic functions



LIMITS OF TRANSCENDENTAL FUNCTIONS



LIMITS OF TRIGONOMETRIC FUNCTION

If a is any number in the domain of corresponding trigonometric functions

1. $\lim_{x\to a} \sin(x) = \sin(a).$
2. $\lim_{x\to a} \cos(x) = \cos(a).$
3. $\lim_{x\to a} \tan(x) = \tan(a).$
4. $\lim_{x\to a} \csc(x) = \csc(a).$
5. $\lim_{x\to a} \sec(x) = \sec(a).$
6. $\lim_{x\to a} \cot(x) = \cot(a).$

LIMITS OF TRIGONOMETRIC FUNCTION

Limit as x→±∞:

$$\lim_{x\to\pm\infty} \sin(x) \qquad \lim_{x\to\pm\infty} \cos(x) \qquad \text{Limits Do Not Exist}$$

$$\lim_{x\to\pm\infty} \tan(x) \qquad \lim_{x\to\pm\infty} \cot(x) \qquad \text{Limits Do Not Exist}$$

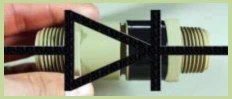
$$\lim_{x\to\pm\infty} \sec(x) \qquad \lim_{x\to\pm\infty} \csc(x) \qquad \text{Limits Do Not Exist}$$

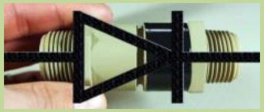
LIMITS OF TRANSCENDENTAL FUNCTIONS

LIMITS OF TRIGONOMETRIC FUNCTION

SPECIAL LIMITS

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$





LIMITS OF TRIGONOMETRIC FUNCTION

Squeeze Theorem.

Suppose we have an inequality of functions

$$g(x) \le f(x) \le h(x)$$

in an interval around c. Then $$\lim_{x \to c} g(x) \le \lim_{x \to c} f(x) \le \lim_{x \to c} h(x)$$

provided those limits exist.

When the limits on the upper bound and lower bound are the same, then the function in the middle is “squeezed” into having the same limit.

LIMITS OF TRIGONOMETRIC FUNCTION

Squeeze Theorem.

Suppose we have an inequality of functions $g(x) \leq f(x) \leq h(x)$

in a interval around c and that $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x).$

Then $\lim_{x \to c} f(x) = L.$

LIMITS OF TRANSCENDENTAL FUNCTIONS

Example

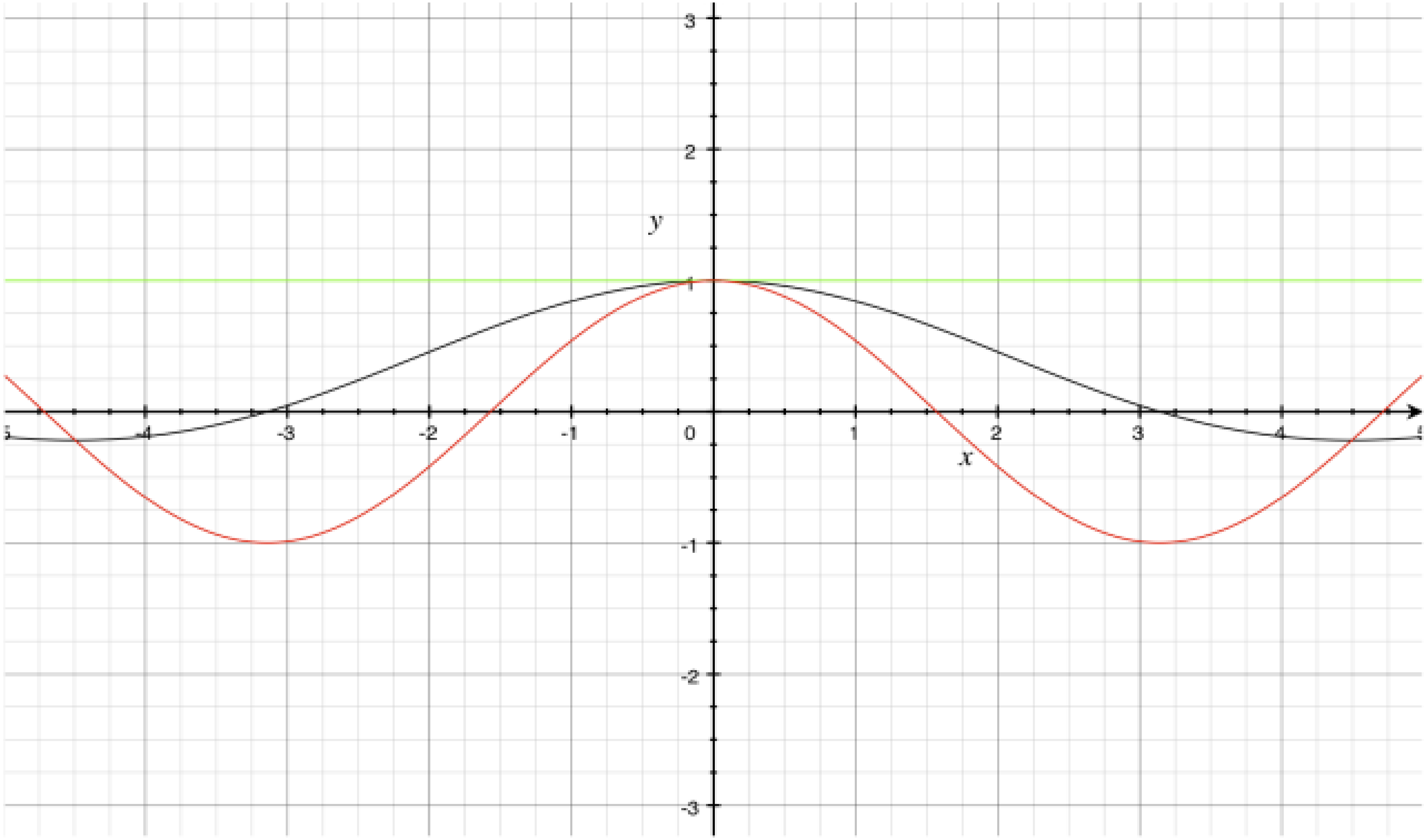
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

To do this, we'll use the Squeeze theorem by establishing upper and lower bounds on sin(x)/x in an interval around 0. Specifically, we'll show that

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

in an interval around 0.

LIMITS OF TRANSCENDENTAL FUNCTIONS



$y = \cos(x)$ $\qquad$ $y = \dfrac{\sin(x)}{x}$ $\qquad$ $y = 1$

We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 1.7.1. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$.Later we will show that we can also consider $\theta \leq 0$.)

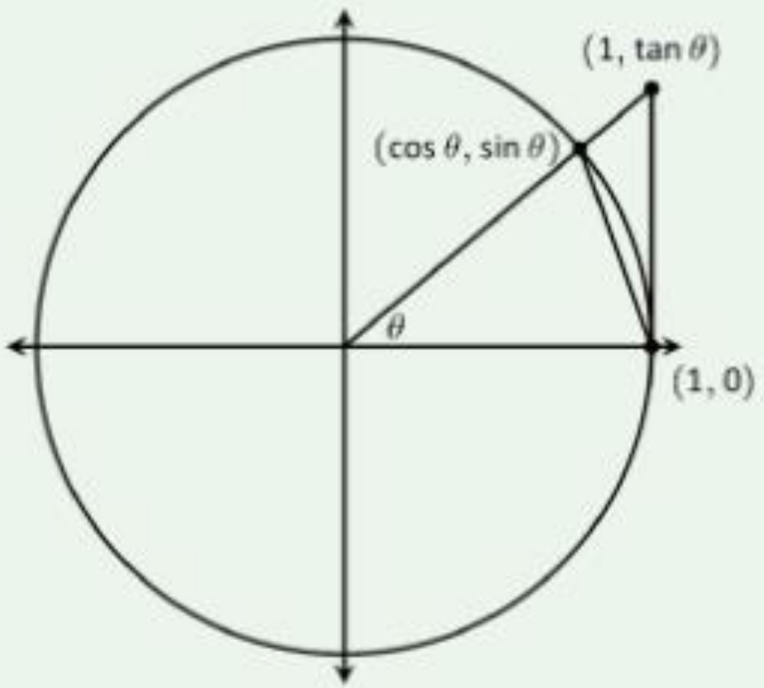
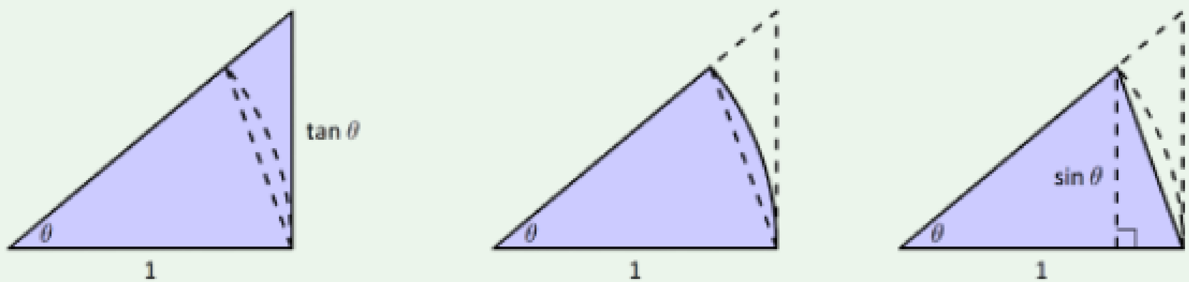


Figure 1.7.1: *The unit circle and related triangles.*

LIMITS OF TRANSCENDENTAL FUNCTIONS

Figure 1.19 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply all terms by $\frac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

LIMITS OF TRANSCENDENTAL FUNCTIONS

Now take limits.

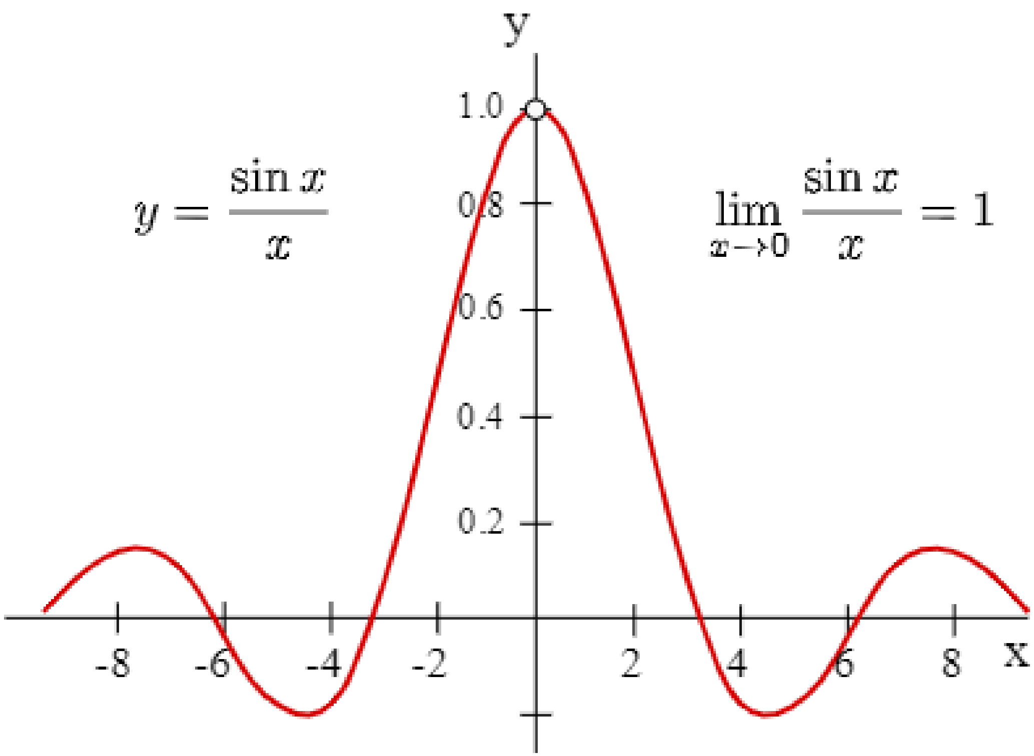
$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le \lim_{\theta\to 0} 1$$

$$\cos 0 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

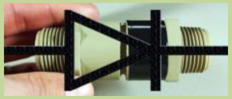
$$1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

Clearly this means that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

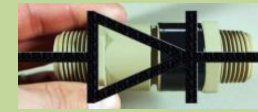
Similarly, If we take a look at the graph of sin x/x



Notice that $x = 0$ is not in the domain of this function. Nevertheless, we can look at the limit as x approaches 0. From the graph we find that the limit is 1 (there is an open circle at $x = 0$ indicating 0 is not in the domain).



LIMITS OF TRANSCENDENTAL FUNCTIONS



Evaluate $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$.

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta \to 0} \frac{1-cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{sin^2\theta}{\theta(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta}$$

$$= 1 \cdot \frac{0}{2} = 0.$$

Therefore,

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0.$$

Other examples

a) $\lim_{x\to 0} \frac{\sin(4x)}{x}$ $\quad \frac{0}{0}$ type

$$= \frac{\sin(4x)}{x} \blacksquare \frac{4}{4}$$

$$= 4 \blacksquare \frac{\sin(4x)}{4x}$$

Since these form is $\frac{\sin(x)}{x} = 1$

$$= 4 \blacksquare 1$$

$$\lim_{x\to 0} \frac{\sin(4x)}{x} = \mathbf{4}$$

b) $\lim_{x\to 0} \frac{\tan x}{x}$

$$= \frac{\frac{\sin(x)}{\cos x}}{x}$$

$$= \frac{\sin x}{x \cos x}$$

$$= \frac{\sin x}{x} \blacksquare \frac{1}{\cos x}$$

$$= 1 \blacksquare \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\cos(0)} = 1$$

LIMITS OF TRANSCENDENTAL FUNCTIONS

c) $\lim\limits_{x \to 0} \dfrac{\sec x - 1}{x}$.

from reciprocal identities
sec x = 1/cos x

$$\lim_{x \to 0} \frac{\sec x - 1}{x} = \lim_{x \to 0} \frac{\dfrac{1}{\cos x} - 1}{x}$$

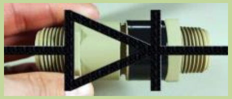
$$= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x}$$

$$= \lim_{x \to 0} \left(\frac{1}{\cos x}\right) \cdot \left(\frac{1 - \cos x}{x}\right)$$

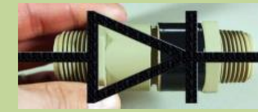
$$= \left[\lim_{x \to 0} \frac{1}{\cos x}\right] \cdot \left[\lim_{x \to 0} \frac{1 - \cos x}{x}\right]$$

$$= 1 \cdot 0$$

Special limit



LIMITS OF TRANSCENDENTAL FUNCTIONS



d) $\lim_{x \to 0} sin\left(\frac{x^2-1}{x-1}\right)$

e) $\lim_{x \to 0} \frac{\sin(2x)}{\sin(5x)}$

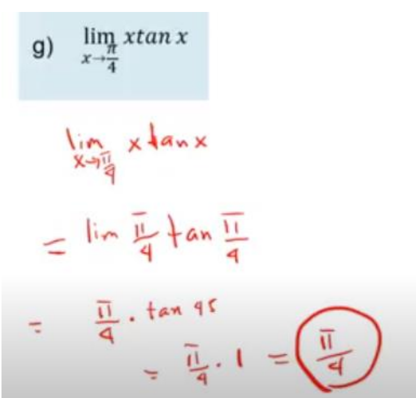
f)

2. $\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0}\left(\frac{\sin 2x}{x} \bullet \frac{2}{2}\right)$

$= 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= (2)(1) = 2$

g) $\lim_{x \to \frac{\pi}{4}} x \tan x$



3. $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0}\left(\frac{\frac{\sin 3x}{x}}{\frac{\sin 5x}{x}}\right)$

$= \lim_{x \to 0} \frac{3 \bullet \frac{\sin 3x}{3x}}{5 \bullet \frac{\sin 5x}{5x}}$

$= \frac{3 \bullet 1}{5 \bullet 1} = \frac{3}{5}$

LIMITS OF TRANSCENDENTAL FUNCTIONS

$$\lim_{t\to 0}\frac{1-\cos t}{\sin t}=\lim_{t\to 0}\frac{\dfrac{1-\cos t}{t}}{\dfrac{\sin t}{t}}=\frac{\displaystyle\lim_{t\to 0}\frac{1-\cos t}{t}}{\displaystyle\lim_{t\to 0}\frac{\sin t}{t}}=\frac{0}{1}=0$$

$$\lim_{x\to 0}\frac{\sin 4x}{\tan x}=\lim_{x\to 0}\frac{\dfrac{4\sin 4x}{4x}}{\dfrac{\sin x}{x\cos x}}$$

$$=\frac{\displaystyle 4\lim_{x\to 0}\frac{\sin 4x}{4x}}{\left(\displaystyle\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\displaystyle\lim_{x\to 0}\frac{1}{\cos x}\right)}=\frac{4}{1\cdot 1}=4$$

LIMITS OF TRANSCENDENTAL FUNCTIONS

a) $\lim_{x\to 0} \dfrac{\sin(4x)}{x}$ $= \dfrac{\sin 4(0)}{0} = \dfrac{0}{0}$ not allowed

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

$= \dfrac{\sin(4x)}{x} \cdot \dfrac{4}{4}$

$\dfrac{\sin(2x)}{2x}$

$= \dfrac{4\sin(4x)}{4x}$

$\dfrac{\sin 3x}{3x}$

$= \lim_{x\to 0} 4 \cdot \lim_{x\to 0} \dfrac{\sin 4x}{4x}$

$= 4 \cdot 1 = 4$

b) $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$ $= \dfrac{\quad}{0}$

$$= \frac{\dfrac{\sin x}{\cos x}}{x}$$

$$= \frac{\sin x}{x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos 0} = 1 \cdot 1 = 1$$

c) $\lim_{x \to 0} \frac{\sec x - 1}{x}$.

Use reciprocal identity for sec x

$\sec = \frac{1}{\cos x}$

$= \frac{\frac{1}{\cos x} - \frac{1}{1}}{x}$

$= \frac{\frac{1 - \cos x}{\cos x}}{x}$

$= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x}$

$= \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = (1) \cdot \frac{1}{\cos 0}$

$= 0 \cdot 1 = 0$

$\frac{\sin x}{x} = 1$

$\frac{1 - \cos x}{x} = 0$

=

$\lim_{x\to 0} sin(\frac{x^2-1}{x-1})$

$\sin\left(\frac{0^2-1}{0-1}\right)$

$\sin\left(\frac{-1}{-1}\right)$

$\boxed{\sin 1}$

e) $\lim_{x\to 0} \frac{\sin(2x)}{\sin(5x)}$

$$= \frac{\sin 2x \cdot \frac{2x}{2x}}{\sin 5x \cdot \frac{5x}{5x}}$$

$$= \frac{2x \frac{\sin 2x}{2x}}{5x \frac{\sin 5x}{5x}} = \frac{\lim_{x\to 0} 2x \cdot \lim_{x\to 0} \frac{\sin 2x}{2x}}{\lim_{x\to 0} 5x \cdot \lim_{x\to 0} \frac{\sin 5x}{5x}} = \frac{2}{5} \cdot \frac{1}{1}$$

$$\boxed{= \frac{2}{5}}$$

Assignment/Activity 2

1. $\lim_{x \to 0} \frac{\cos x}{x+1}$

2. $\lim_{\theta \to \pi/2} \theta \cos \theta$

3. $\lim_{t \to 0} \frac{\cos^2 t}{1+\sin t}$

4. $\lim_{x \to 0} \frac{3x \tan x}{\sin x}$

5. $\lim_{x \to 0} \frac{\sin x}{2x}$

6. $\lim_{\theta \to 0} \frac{\sin 3\theta}{2\theta}$

7. $\lim_{x \to 0} \frac{\sin(8x)}{x}$

8. $\lim_{\theta \to \pi/2} \frac{\cos^2(\theta)}{1-\sin(\theta)}$.

9. $\lim_{x \to 0} (\sin^2 x + \cos^2 x) - \cos x)\frac{1}{x}$

10. $\lim_{x \to \frac{5\pi}{6}} 3 \tan x$

Solution to Act. 2

1. $\lim_{x\to 0}\frac{\cos x}{x+1} = \frac{\cos 0}{0+1} = \frac{1}{1} = \boxed{1}$

2. $\lim_{\theta\to \pi/2}\theta\cos\theta = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = \boxed{0}$

3. $\lim_{t\to 0}\frac{\cos^2 t}{1+\sin t} = \frac{(\cos 0)^2}{1+\sin 0} = \frac{1}{1+0} = \boxed{1}$

4. $\lim_{x\to 0}\frac{3x\tan x}{\sin x} = \frac{3(0)\tan 0}{\sin 0} = \frac{0}{0}$

$$= \lim_{x\to 0}\frac{3x\,\frac{\sin x}{\cos x}}{\sin x}$$

$$= \lim_{x\to 0}\frac{3x\sin x}{\sin x\cos x}$$

$$= \lim_{x\to 0}\frac{3x}{\cos x}$$

$$= \frac{3(0)}{\cos 0} = \frac{0}{1} = \boxed{0}$$

5. $\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \boxed{\frac{1}{2}}$

6. $\lim_{\theta\to 0} \frac{\sin 3\theta}{2\theta} = \lim_{\theta\to 0} \left(\frac{1}{2} \cdot \frac{\sin 3\theta}{\theta}\right) \frac{3}{3}$

$= \lim_{\theta\to 0} \frac{1}{2} \cdot \frac{3\sin 3\theta}{3\theta}$

$= \lim_{\theta\to 0} \frac{1}{2} \cdot 3 \cdot \frac{\sin 3\theta}{3\theta}$

$= \lim_{\theta\to 0} \frac{1}{2} \cdot 3 \cdot 1$

$\boxed{= \frac{3}{2}}$

7. $\lim_{x\to 0} \dfrac{\sin(8x)}{x}$ $= \lim_{x\to 0} \dfrac{\sin 8x}{x} \cdot \dfrac{8}{8} = \lim_{x\to 0} 8 \dfrac{\sin 8x}{8x} = 8 \cdot 1 = \boxed{8}$

8. $\lim_{\theta\to\pi/2} \dfrac{\cos^2(\theta)}{1-\sin(\theta)}$

From trigonometric identity $\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$= \lim_{\theta\to\frac{\pi}{2}} \dfrac{1-\sin^2(\theta)}{1-\sin(\theta)} = \lim_{\theta\to\frac{\pi}{2}} \dfrac{(1+\sin(\theta))(1-\sin(\theta))}{1-\sin(\theta)} = \lim_{\theta\to\frac{\pi}{2}} 1+\sin\theta = 1+\sin\frac{\pi}{2} = 1+1 = \boxed{2}$

9. $\lim_{x \to 0} (\sin^2 x + \cos^2 x) - \cos x)\frac{1}{x}$ $= \lim_{x \to 0} \frac{(1 - \cos x)}{x} = \boxed{0}$

10. $\lim_{x \to \frac{5\pi}{6}} 3 \tan x$

$= 3 \tan \frac{\pi}{6}$

$= 3 \cdot \frac{-\sqrt{3}}{3}$

$\boxed{= -\sqrt{3}}$

$$\sin x, \cos x, \tan x = \frac{\sin x}{\cos x}, \csc x = \frac{1}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

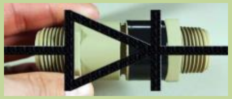
The following are some properties of these functions:

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

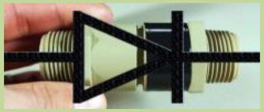
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \sin(-x) = -\sin x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \cos(-x) = \cos x$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \qquad \tan(-x) = -\tan x$$



LIMITS OF TRANSCENDENTAL FUNCTIONS



LIMITS EXPONENTIAL FUNCTION

Power Rule

$$\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} f(x)^{\lim_{x \to a} g(x)}$$

It is a property of power rule, used to find the limit of an exponential function whose base and exponent are in a function form..

Constant Base Power Rule

Formula

$$\lim_{x \to a} b^{f(x)} = b^{\lim_{x \to a} f(x)}$$

The limit of an exponential function is equal to the limit of the exponent with same base. It is called the limit rule of an exponential function.

LIMITS OF TRANSCENDENTAL FUNCTIONS

Constant Exponent Power Rule

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$$

Example

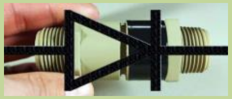
Evaluate $\lim\limits_{x \to 1} \left(3^{x+2}\right)$

Find the value of the given function by direct substitution.

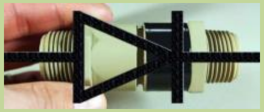
$$\Longrightarrow \lim_{x \to 1} \left(3^{x+2}\right) = 3^{1+2}$$

$$\Longrightarrow \lim_{x \to 1} \left(3^{x+2}\right) = 3^{3}$$

$$\Longrightarrow \lim_{x \to 1} \left(3^{x+2}\right) = 27$$



LIMITS OF TRANSCENDENTAL FUNCTIONS



2. $\lim_{x \to 2} (2^{4-x^2})$

$= 2^{(4-2^2)}$

$= 2^{(0)}$

$= 1$

3. $\lim_{x \to -2} [\frac{(3 - x^2)}{4^{-2x}}]$

$= \frac{3-(-2)^{2)}}{4^{(-2)(-2)}}$

$= \frac{3-4}{256}$

$= -\frac{1}{256}$

LIMITS OF TRANSCENDENTAL FUNCTIONS

5. $\lim\limits_{x \to 3} (2^{4-x^2})$

6. $\lim\limits_{x \to 3} \frac{(2^{4-x^2})}{2^{x-5}})$

LIMITS OF TRANSCENDENTAL FUNCTIONS

5. $\lim_{x\to 3}(2^{4-x^2})$

$= 2^{4-(3)^2}$

$= 2^{4-9}$

$= 2^{-5}$

$= \frac{1}{2^5}$

$= \frac{1}{32}$

6. $\lim_{x\to 3}\frac{(2^{4-x^2})}{2^{x-5}})$

$\frac{2^{4-9}}{2^{3-5}}$

$= \frac{2^{-5}}{2^{-2}}$

$= \frac{1}{2^{5-2}} = \frac{1}{2^3} = \frac{1}{8}$

$\rightarrow \frac{2^{2}}{2^{5}} = \frac{4}{32}$ or $\frac{1}{8}$

LIMITS OF TRANSCENDENTAL FUNCTIONS

Special limits

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

LIMITS OF TRANSCENDENTAL FUNCTIONS

1. $\lim\limits_{x\to 0} \frac{(5e^{x}-5)}{2x})$

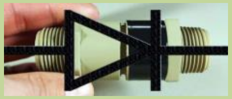
$= \lim\limits_{x\to 0} \frac{5(e^{x}-1)}{2\,x})$

$= \lim\limits_{x\to 0} \; \frac{5}{2} \blacksquare \left(\frac{e^{x}-1}{x}\right)$

$= \lim\limits_{x\to 0} \; \frac{5}{2} \blacksquare \; \lim\limits_{x\to 0}\left(\frac{e^{x}-1}{x}\right)$

$= \frac{5}{2} \; \blacksquare \; 1$

$= \frac{5}{2}$



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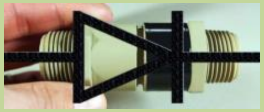


LIMITS OF LOGARITHMIC FUNCTIONS

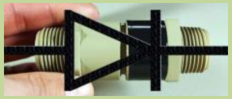
1. $\lim_{x \to c} \log_b x = \log_b c$
2. $\lim_{x \to c} \log_b f(x) = \log_b f(c), \ f(c) > 0$

when b >1

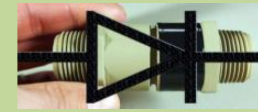
$\lim_{x \to \infty} \log_b x = \infty$ $\lim_{x \to 0} \log_b x = -\infty$

when 0< b <1

$\lim_{x \to \infty} \log_b x = -\infty$ $\lim_{x \to 0} \log_b x = \infty$



LIMITS OF TRANSCENDENTAL FUNCTIONS



Examples

1. $\lim_{x \to 2} log\ x \qquad = \log 2$

2. $\lim_{x \to -5} (\log_5 25x^2)$

$= \log_5 25\,(-5^2)$

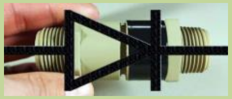
$= \log_5 625$

$= 4$

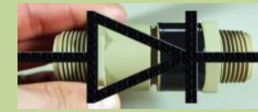
3. $\lim_{x \to 5} \log_2(\,12x + 4)$

$= \log_2 64$

$= 6$



LIMITS OF TRANSCENDENTAL FUNCTIONS



Examples

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LIMITS OF TRANSCENDENTAL FUNCTIONS

Examples

4. $\lim_{x \to 2} log(x^4 - 8)$

5. $\lim_{x \to 2} \log_2(12x + 8)$

Examples

4. $\lim_{x \to 2} log(x^4 - 8)$

$= \lim_{x \to 2} \log(2^4 - 8)$

$= \log 8$

5. $\lim_{x \to 2} \log_2(12x + 8)$

$= \lim \log_2(24 + 8)$

$= \log_2 32$

$= 5$

$32 = 2^y$

$2^5 = 2^y$

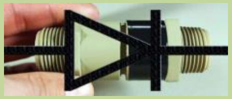
$5 = y$

LIMITS OF TRANSCENDENTAL FUNCTIONS

6)

$$\lim_{x\to 0} \log_2 \frac{256-x^2}{x+2} = \log_2 \frac{256}{2} = \log_2 128$$

$$= 7$$



table

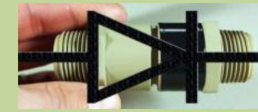
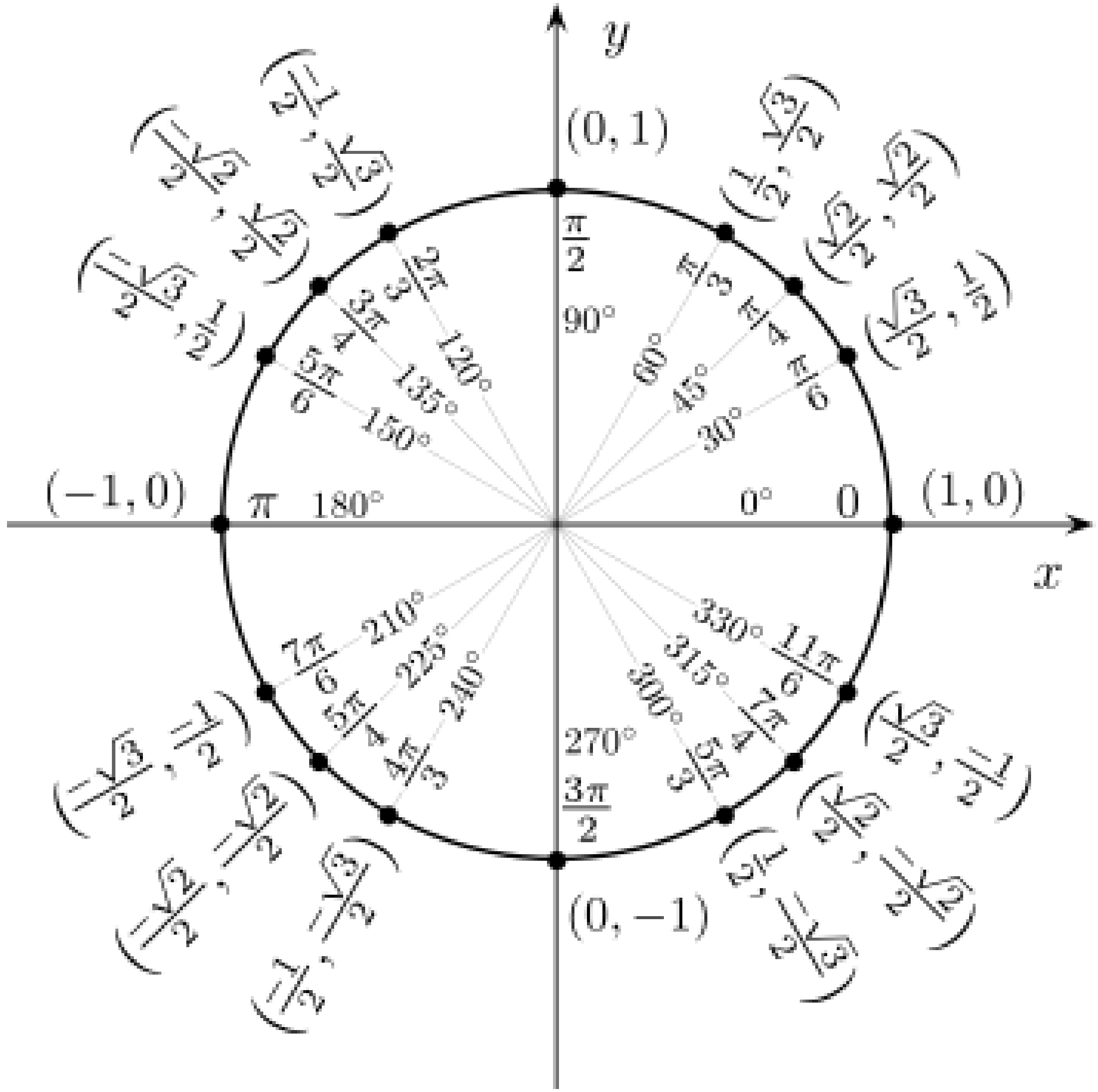


TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

table



y
x
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
0°
0
30°
π/6
45°
π/4
60°
π/3
90°
π/2
120°
2π/3
135°
3π/4
150°
5π/6
180°
π
210°
7π/6
225°
5π/4
240°
4π/3
270°
3π/2
300°
5π/3
315°
7π/4
330°
11π/6
(√3/2, 1/2)
(√2/2, √2/2)
(1/2, √3/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
(−1/2, −√3/2)
(1/2, −√3/2)
(√2/2, −√2/2)
(√3/2, −1/2)