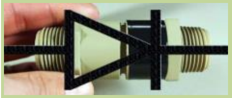
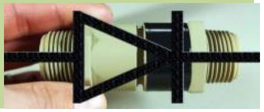
CALCULUS 1

LIMITS OF A FUNCTION

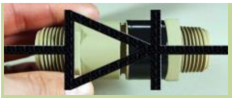
PREPARED BY ENGR. JOHN R. REJANO,ECE



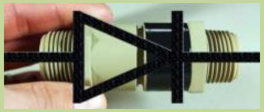
OBJECTIVES



- Discuss the limits of functions
- Differentiate left and right limits
- Discus theorem on limits



LIMITS

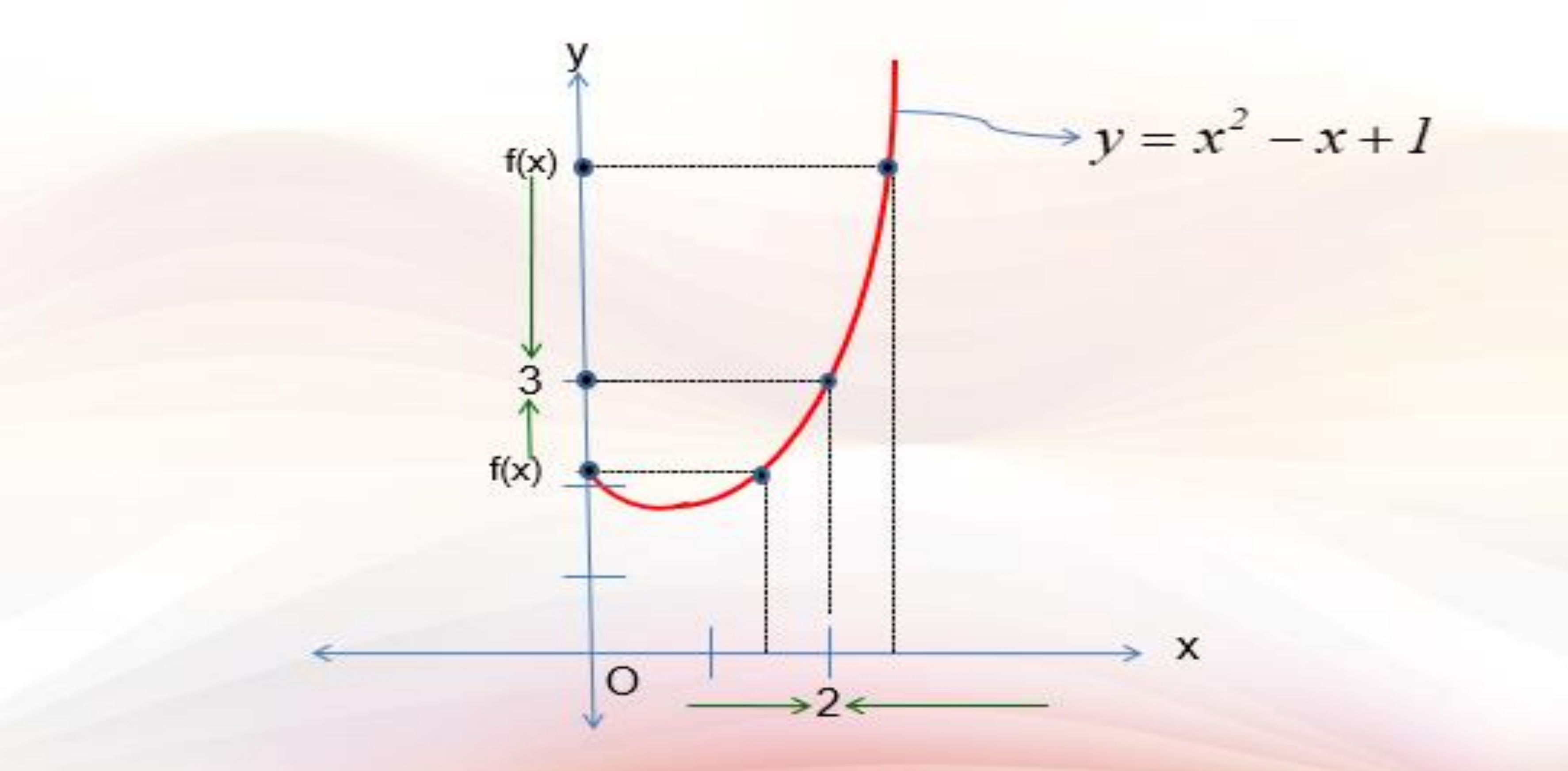


DEFINITION: LIMITS

- The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value.

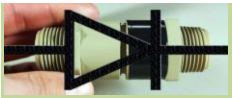
- For example let us examine the behavior of the function
  $f(x) = x^2$- x +1 for x-values closer and closer to 2.

LIMITS

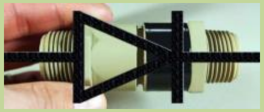


x	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1
F(x)	2.71	2.852	2.97	2.985	2.997		3.003	3.015	3.031	3.152	3.31

left side → ← right side



LIMITS



We describe this by saying that the "limit of f$(x) = x^2$- x +1 is 3 as x approaches 2 from either side,"

Thus;

we write

$$\lim_{x \to 2} \left(x^2 - x + 1 \right) = 3$$

LIMITS

Limit of a Function

- If *f* is a function, then we say:
  A is the limit of f (*x*) *as x approaches a*

if the value of *f* (*x*) gets arbitrarily close to *A* as *x* approaches *a*. This is written in mathematical notation as:

$$\lim_{x \to a} f(x) = A$$

LIMITS

For example, $\lim_{x\to 3} x^2 = 9$, since x^2 gets arbitrarily close to 9 as x approaches as close as one wishes to 3. The definition of $\lim_{x\to a} f(x) = A$ was stated above in ordinary language. The definition can be stated in more precise mathematical language as follows: $\lim_{x\to a} f(x) = A$ if and only if, for any given positive number ϵ, however small, there exists a positive number δ such that, whenever $0 < |x - a| < \delta$, then $|f(x) - A| < \epsilon$.

The gist of the definition is illustrated in Fig. 7-1. After ϵ has been chosen [that is, after interval (ii) has been chosen], then δ can be found [that is, interval (i) can be determined] so that, whenever $x \neq a$ is on interval (i), say at x_0, then $f(x)$ is on interval (ii), at $f(x_0)$. Notice the important fact that whether or not $\lim_{x\to a} f(x) = A$ is true does not depend upon the value of $f(x)$ when $x = a$. In fact, $f(x)$ need not even be defined when $x = a$.

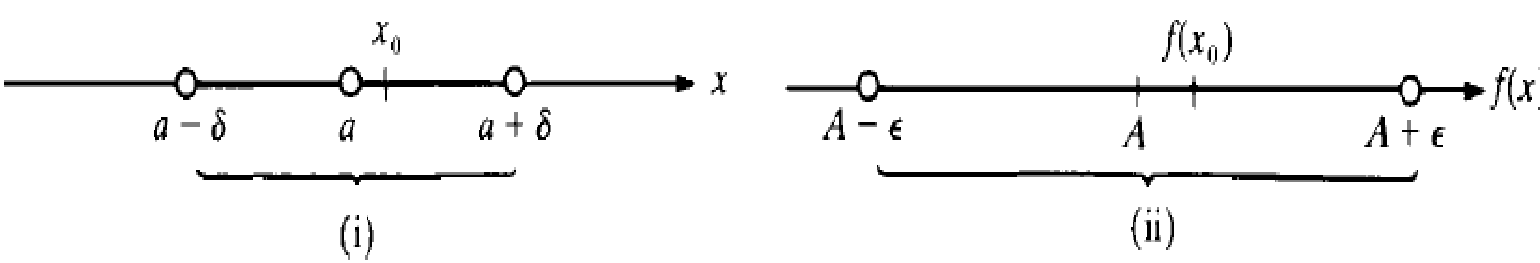


Fig. 7-1

LIMITS

EXAMPLE 7.1: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, although $\frac{x^2 - 4}{x - 2}$ is not defined when $x = 2$. Since

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

we see that $\frac{x^2 - 4}{x - 2}$ approaches 4 as x approaches 2.

LIMITS

Right and Left Limits

- By $\lim_{x \to a^-} f(x) = A$ we mean that, f is defined in some

  open interval (c, a) and f (x) approaches A as x approaches a through values less than a, that is, *as x approaches a from the left*

- Similarly, $\lim_{x \to a^+} f(x) = A$

  means that, *f* is defined in some open interval (*a, d*) and *f* (*x*) approaches *A* as *x* approaches *a from the right*.

LIMITS

- If *f* is defined in an interval to the left of *a* and in an interval to the right of *a*, then the statement

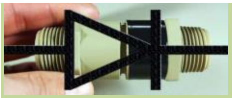
$$\lim_{x \to a} f(x) = A$$

is equivalent to the conjunction of the two statements

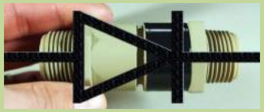
$$\lim_{x \to a^-} f(x) = A \text{ and } \lim_{x \to a^+} f(x) = A.$$

- When a function is defined only on one side of a point *a*, then we shall identify $\lim_{x \to a} f(x)$

with the one-sided limit, if it exists.



LIMITS



If f is a real valued function, then x can approach a from two sides: the left side of a and the right side of a. This is illustrated with the help of the diagram below.

Left Hand Limit

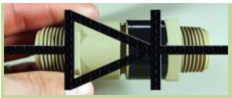
If ***x*** *approaches* ***a*** from the left side, i.e. from the values less than a, the function is said to have a left hand limit. If A is the left hand limit of f as x approaches a, we write it as

$$\lim_{x \to a^-} f(x) = A$$

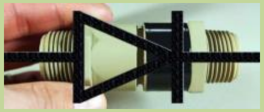
Right Hand Limit

If ***x approaches a*** from the right side, i.e. from the values greater than a, the function is said to have a right hand limit. If A is the right hand limit of f as x approaches a, we write it as

$$\lim_{x \to a^+} f(x) = A$$



LIMITS



For the existence of the limit of a real valued function at a certain point, it is essential that both its left hand and right hand limits exist and have the same value.
In other words, if the left and right hand limits exist and

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

then f is said to have a limit at x=a.

On the other hand if both the left and right hand limits exist but

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

then the limit of f does not exist at x=a.

LIMITS

Example:

if $f(x) = \sqrt{x}$

then *f* is defined only to the right of 0. Hence, since

$$\lim_{x \to 0^+} \sqrt{x} = 0, \text{ we will also write } \lim_{x \to 0} \sqrt{x} = 0.$$

Of course, $\lim_{x \to 0^-} \sqrt{x}$ does not exist, since $\sqrt{x}$ is not defined when $x < 0$.

This is an example where the existence of the limit from one side does not entail the existence of the limit from the other side.

LIMITS

EXAMPLE 7.3: The function $f(x) = \sqrt{9 - x^2}$

has the interval $-3 \le x \le 3$ as its domain.

If *a* is any number on the interval (−3, 3), then

$\lim_{x \to a} \sqrt{9 - x^2}$ exists and is equal to $\sqrt{9 - a^2}$.

Now consider *a* = 3. Let *x* approach 3 from the left; then

$\lim_{x \to 3^-} \sqrt{9 - x^2} = 0$. For $x > 3$, $\sqrt{9 - x^2}$ is not defined, since $9 - x^2$ is negative.

Hence, $\lim_{x \to 3} \sqrt{9 - x^2} = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0.$

Similarly, $\lim_{x \to -3} \sqrt{9 - x^2} = \lim_{x \to -3^+} \sqrt{9 - x^2} = 0.$

LIMITS

Theorems on Limits

Theorem 7.1: If $f(x) = c$, a constant, then $\lim_{x \to a} f(x) = c$.

For the next five theorems, assume $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$.

Theorem 7.2: $\lim_{x \to a} c \cdot f(x) = c \lim_{x \to a} f(x) = cA.$

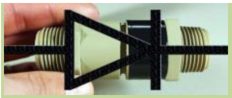
Theorem 7.3: $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B.$

LIMITS

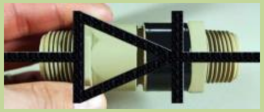
Theorem 7.4: $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B.$

Theorem 7.5: $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}, \text{ if B} \neq 0.$

Theorem 7.6: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}, \text{ if } \sqrt[n]{A} \text{ is defined.}$



LIMITS



Examples:

1. $\lim_{x \to a} 4 = 4$, $\lim_{x \to a} 3.14 = ?$ (7.1)

2. $\lim_{x \to 2} 5x =$ (7.2)

3. $\lim_{x \to 2} (2x + 3) =$ (7.3)

$\lim_{x \to 2} (x^2 - 4x + 1) =$

4. $\lim_{x \to 1/2} (4x)(3x) =$ (7.4)

LIMITS

1) $\lim_{x \to 2} 5x = 5 \lim_{x \to 2} x = 5 \cdot 2 = 10$

3) $\lim_{x \to 2} (2x+3) = 2 \lim_{x \to 2} x + \lim_{x \to 2} 3$

$= 2 \cdot 2 + 3$

$= 4 + 3$

$= 7$

$\lim_{x \to 2} (x^2 - 4x + 1) = \lim_{x \to 2} x^2 - 4 \lim_{x \to 2} x + \lim_{x \to 2} 1$

$= 2^2 - (4 \cdot 2) + 1$

$= 4 - 8 + 1$

$= -3$

LIMITS

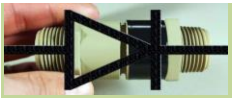
4) $\lim_{x \to \frac{3}{2}} (4x)(3x) = \lim_{x \to \frac{1}{2}} 4x \cdot \lim_{x \to \frac{1}{2}} 3x$

$= 4\left(\frac{1}{2}\right) \cdot 3\left(\frac{1}{2}\right)$

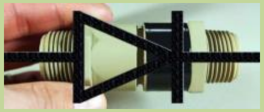
$= 2 \cdot \frac{3}{2}$

$= \frac{6}{2}$

$= 3$



LIMITS



5. $$\lim_{x \to 3} \frac{x-2}{x+2}$$ (7.5)

6. $$\lim_{x \to 4} \sqrt{25 - x^2} =$$ (7.6)

7. $$\lim_{x \to 4} \frac{x-4}{x^2 - x - 12}$$

Note: not allowed
- 0/0
- 0 in the denominator

LIMITS

5. $\lim\limits_{x\to 3}\dfrac{x-2}{x+2}$ $= \dfrac{\lim\limits_{x\to 3} x-2}{\lim\limits_{x\to 3} x+2} = \dfrac{3-2}{3+2} = \dfrac{1}{5}$

6. $\lim\limits_{x\to 4}\sqrt{25-x^2} =$ $\sqrt{\lim\limits_{x\to 4} 25-x^2} = \sqrt{25-4^2} = \sqrt{25-16} = \sqrt{9} = 3$

7. $\lim\limits_{x\to 4}\dfrac{x-4}{x^2-x-12}$ $= \dfrac{4-4}{4^2-4-12} = \dfrac{0}{0}$ indeterminate (not allowed)

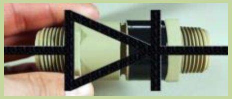
Factor then take the limit

$$\lim_{x\to 4}\frac{(x-4)}{(x-4)(x+3)} = \lim_{x\to 4}\frac{1}{x+3} = \frac{1}{4+3} = \frac{1}{7}$$

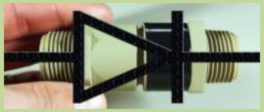
LIMITS OF TRIGONOMETRIC FUNCTION

If a is any number in the domain of corresponding trigonometric functions

1. $\lim_{x \to a} \sin(x) = \sin(a).$
2. $\lim_{x \to a} \cos(x) = \cos(a).$
3. $\lim_{x \to a} \tan(x) = \tan(a).$
4. $\lim_{x \to a} \csc(x) = \csc(a).$
5. $\lim_{x \to a} \sec(x) = \sec(a).$
6. $\lim_{x \to a} \cot(x) = \cot(a).$



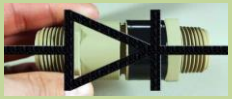
LIMITS OF TRANSCENDENTAL FUNCTIONS

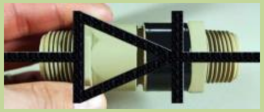


LIMITS OF TRIGONOMETRIC FUNCTION

SPECIAL LIMITS

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$





LIMITS OF TRIGONOMETRIC FUNCTION

Squeeze Theorem.

Suppose we have an inequality of functions

$$g(x) \leq f(x) \leq h(x)$$

in an interval around c. Then $$\lim_{x \to c} g(x) \leq \lim_{x \to c} f(x) \leq \lim_{x \to c} h(x)$$

provided those limits exist.

When the limits on the upper bound and lower bound are the same, then the function in the middle is "squeezed" into having the same limit.

LIMITS OF TRIGONOMETRIC FUNCTION

Squeeze Theorem.

Suppose we have an inequality of functions $g(x) \leq f(x) \leq h(x)$

in a interval around c and that $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x).$

Then $\lim_{x \to c} f(x) = L.$

LIMITS OF TRANSCENDENTAL FUNCTIONS

Example

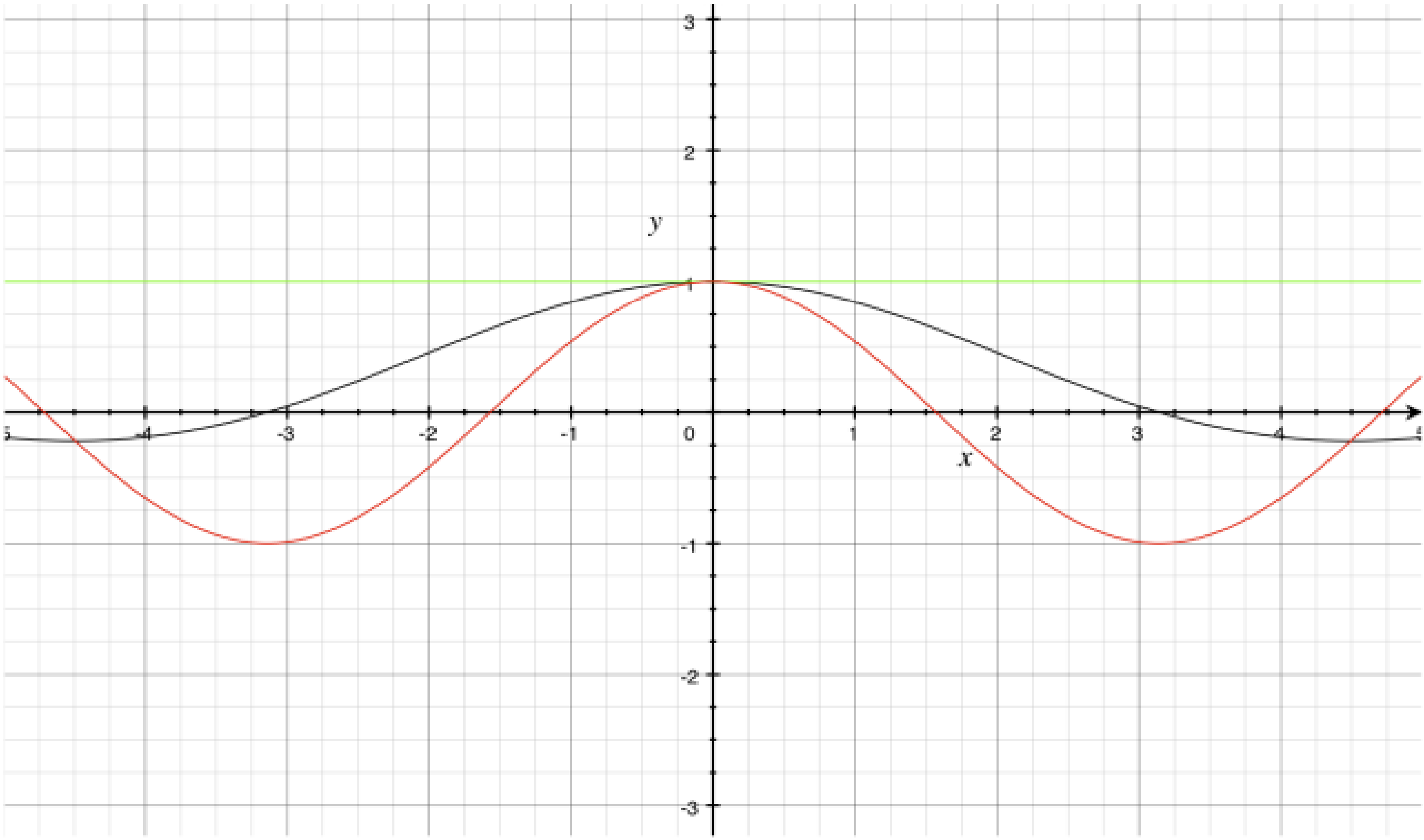
$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1.$$

To do this, we'll use the Squeeze theorem by establishing upper and lower bounds on sin(x)/x in an interval around 0. Specifically, we'll show that

$$\cos(x) \le \frac{\sin(x)}{x} \le 1$$

in an interval around 0.

LIMITS OF TRANSCENDENTAL FUNCTIONS



$y = \cos(x)$ $\qquad$ $y = \dfrac{\sin(x)}{x}$ $\qquad$ $y = 1$

LIMITS OF TRANSCENDENTAL FUNCTIONS

We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 1.7.1. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$.Later we will show that we can also consider $\theta \leq 0$.)

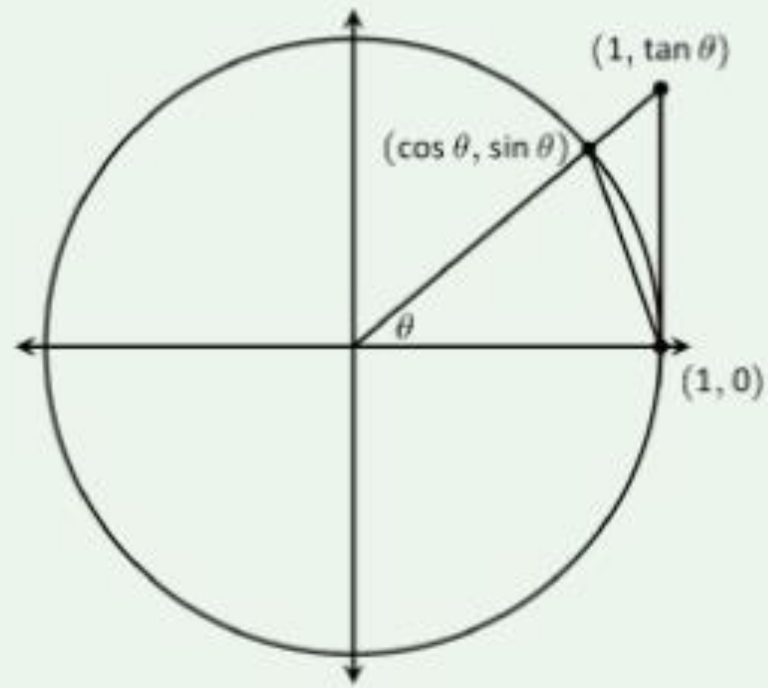
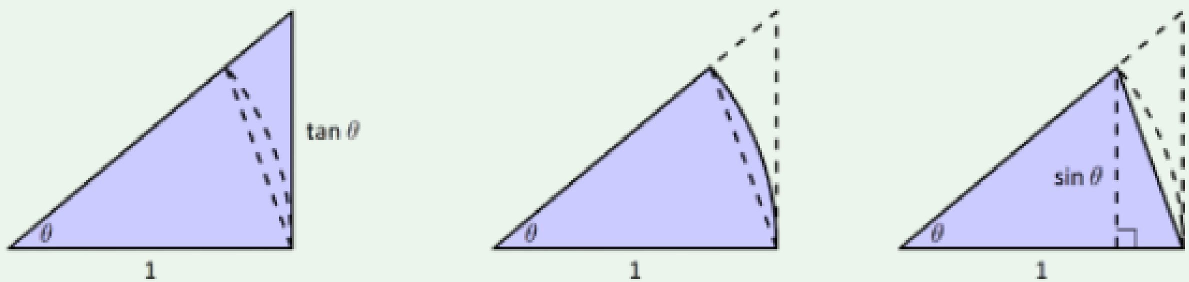


Figure 1.7.1: *The unit circle and related triangles.*

LIMITS OF TRANSCENDENTAL FUNCTIONS

Figure 1.19 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



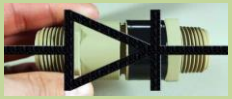
$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply all terms by $\frac{2}{\sin\theta}$, giving

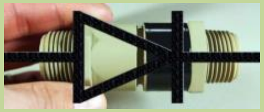
$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$



LIMITS OF TRANSCENDENTAL FUNCTIONS



Now take limits.

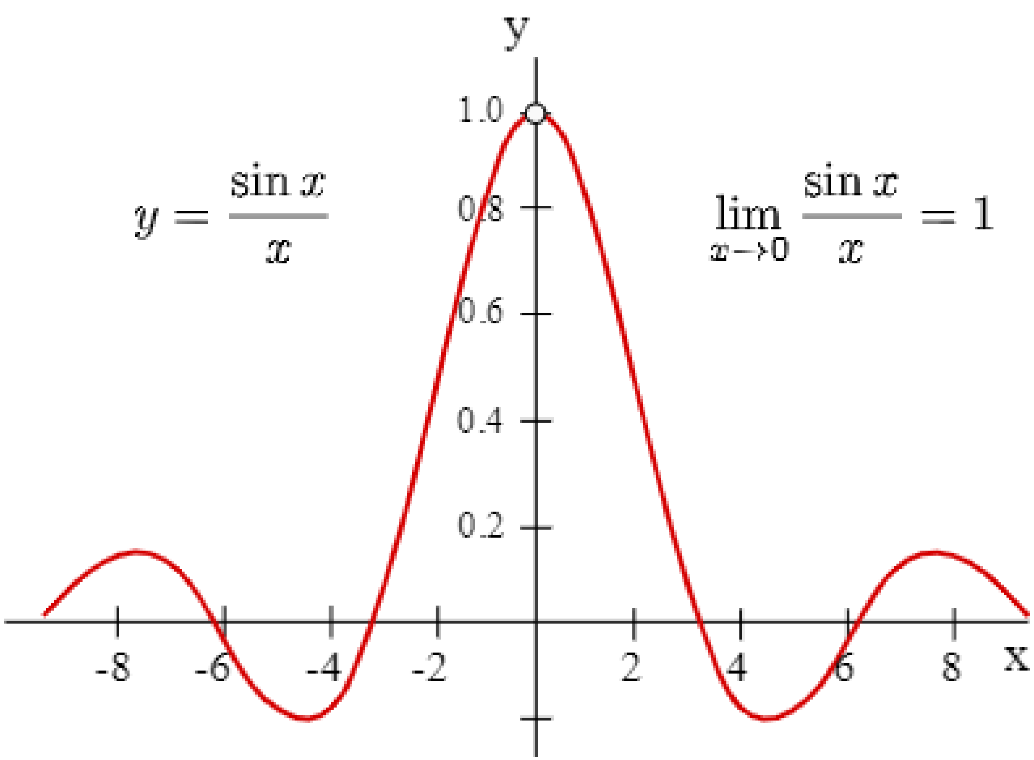
$$\lim_{\theta\to 0} \cos\theta \le \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \le \lim_{\theta\to 0} 1$$

$$\cos 0 \le \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \le 1$$

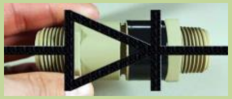
$$1 \le \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \le 1$$

Clearly this means that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

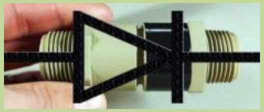
Similarly, If we take a look at the graph of sin x/x



Notice that $x = 0$ is not in the domain of this function. Nevertheless, we can look at the limit as x approaches 0. From the graph we find that the limit is 1 (there is an open circle at $x = 0$ indicating 0 is not in the domain).



LIMITS OF TRANSCENDENTAL FUNCTIONS



Evaluate $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$.

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{1-cos^2\theta}{\theta(1+\cos\theta)}$$

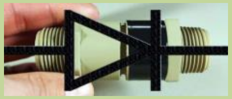
$$= \lim_{\theta\to 0} \frac{sin^2\theta}{\theta(1+\cos\theta)}$$

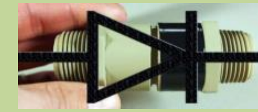
$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta}$$

$$= 1 \cdot \frac{0}{2} = 0.$$

Therefore,

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$$





Other examples

a) $\lim_{x \to 0} \frac{\sin(4x)}{x}$ $\frac{0}{0}$ type

$$= \frac{\sin(4x)}{x} \blacksquare \frac{4}{4}$$

$$= 4 \blacksquare \frac{\sin(4x)}{4x}$$

Since these form is $\frac{\sin(x)}{x} = 1$

$$= 4 \blacksquare 1$$

$$\lim_{x \to 0} \frac{\sin(4x)}{x} = \mathbf{4}$$

b) $\lim_{x \to 0} \frac{\tan x}{x}$

$$= \frac{\frac{\sin(x)}{\cos x}}{x}$$

$$= \frac{\sin x}{x \cos x}$$

$$= \frac{\sin x}{x} \blacksquare \frac{1}{\cos x}$$

$$= 1 \blacksquare \frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{\cos(0)} = 1$$

c) $\lim_{x \to 0} \frac{\sec x - 1}{x}$.

from reciprocal identities
sec x = 1/cos x

$$\lim_{x \to 0} \frac{\sec x - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{\cos x} - 1}{x}$$

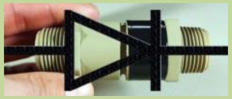
$$= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x}$$

$$= \lim_{x \to 0} \left(\frac{1}{\cos x}\right) \cdot \left(\frac{1 - \cos x}{x}\right)$$

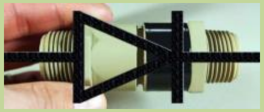
$$= \left[\lim_{x \to 0} \frac{1}{\cos x}\right] \cdot \left[\lim_{x \to 0} \frac{1 - \cos x}{x}\right]$$

$$= 1 \cdot 0$$

Special limit



LIMITS OF TRANSCENDENTAL FUNCTIONS



d) $\lim_{x \to 0} sin\left(\dfrac{x^2-1}{x-1} \right)$

e) $\lim_{x \to 0} \dfrac{\sin(2x)}{\sin(5x)}$

Assignment/Activity 2

1. $\lim\limits_{x \to 0} \dfrac{\cos x}{x+1}$

2. $\lim\limits_{\theta \to \pi/2} \theta \cos \theta$

3. $\lim\limits_{t \to 0} \dfrac{\cos^2 t}{1+\sin t}$

4. $\lim\limits_{x \to 0} \dfrac{3x \tan x}{\sin x}$

5. $\lim\limits_{x \to 0} \dfrac{\sin x}{2x}$

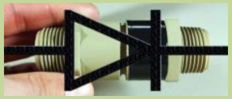
6. $\lim\limits_{\theta \to 0} \dfrac{\sin 3\theta}{2\theta}$

7. $\lim_{x \to 0} \dfrac{\sin(8x)}{x}$

8. $\lim\limits_{\theta \to \pi/2} \dfrac{\cos^2(\theta)}{1-\sin(\theta)}.$

9. $\lim\limits_{x \to 0} (sin^2 x + cos^2 x) - cosx)^{\frac{1}{x}}$

10. $\lim\limits_{x \to \frac{5\pi}{6}} 3 \tan x$



table

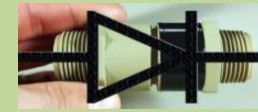
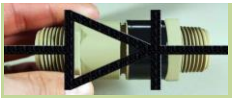
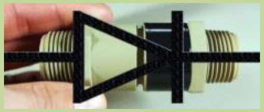


TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0



REFERENCE



1. CALCULUS by H. Anton, et al 10th edition
2. Schaum's outline series CALCULUS 6th edition by Ayres/ Mendelson