ALGEBRA

Exponential and Logarithmic functions

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LEARNING OUTCOMES

- Discuss Exponential and Logarithmic functions
- Graph of an exponential function
- Natural exponential
- Common and natural logarithm
- Converting exponential to logarithmic and vice-versa

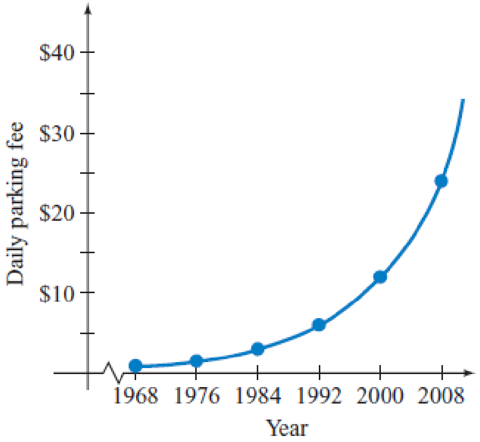

Figure 4.13

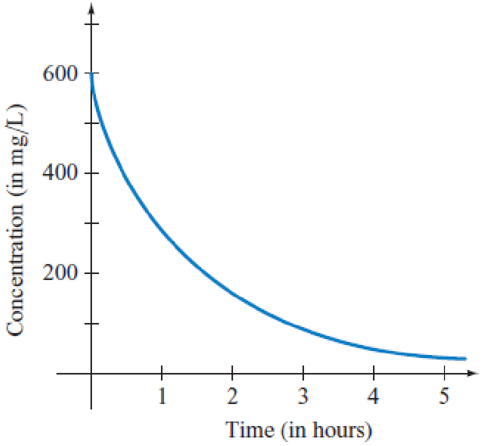

Figure 4.14

Exponential Functions

When an airport parking facility opened in 1968, it charged \$0.75 for all day parking. Since then it has doubled its daily parking fee every 8 years as shown in the following table.

Table 4.1

Year	1968	1976	1984	1992	2000	2008
Daily parking fee	\$0.75	\$1.50	\$3.00	\$6.00	\$12.00	\$24.00

In Figure 4.13, we have plotted the data in the above table and modeled the upward trend in the parking fee by a smooth curve. This model is based on an *exponential function*, which is one of the major topics of this chapter.

The effectiveness of a drug, which is used for sedation during a surgical procedure, depends on the concentration of the drug in the patient. Through natural body chemistry, the amount of this drug in the body decreases over time. The graph in Figure 4.14 models this decrease. This model is another example of an exponential model.

Definition of an Exponential Function

The **exponential function with base b** is defined by

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and x is a real number.

The base b of $f(x) = b^x$ is required to be positive. If the base were a negative number, the value of the function would be a complex number for some values of x. For instance, if

$b = -4$ and $x = \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = (-4)^{1/2} = 2i$. To avoid complex number values of a function, the base of any exponential function must be a positive number. Also, b is defined such that $b \neq 1$ because $f(x) = 1^x = 1$ is a constant function.

You may have noticed that in the definition of an exponential function the exponent x is a real number. We have already worked with expressions of the form b^x, where $b > 0$ and x is a *rational* number. For instance,

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$
$$27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$$
$$32^{0.4} = 32^{2/5} = (\sqrt[5]{32})^2 = 2^2 = 4$$

To extend the meaning of b^x to *real* numbers, we need to give meaning to b^x when x is an *irrational* number. For example, what is the meaning of 5^π? To completely answer this question requires concepts from calculus. However, for our purposes, we can think of 5^π as the unique real number that is approached by 5^x as x takes on ever closer rational number approximations of π. For instance, each successive number in the following list is a closer approximation of 5^π than the number to its left.

$$5^3, 5^{3.1}, 5^{3.14}, 5^{3.142}, 5^{3.1416}, 5^{3.14159}, 5^{3.141593}, 5^{3.1415927}, 5^{3.14159265}, \ldots$$

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EXAMPLE 1 Evaluate an Exponential Function

Evaluate $f(x) = 3^x$ at $x = 2, x = -4$, and $x = \pi$.

Solution

$f(2) = 3^2 = 9$

$f(-4) = 3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

$f(\pi) = 3^\pi \approx 3^{3.1415927} \approx 31.54428$ • Evaluate with the aid of a calculator.

Graphs of Exponential Functions

The graph of $f(x) = 2^x$ is shown in Figure 4.15 on page 348. The coordinates of some of the points on the curve are given in Table 4.2.

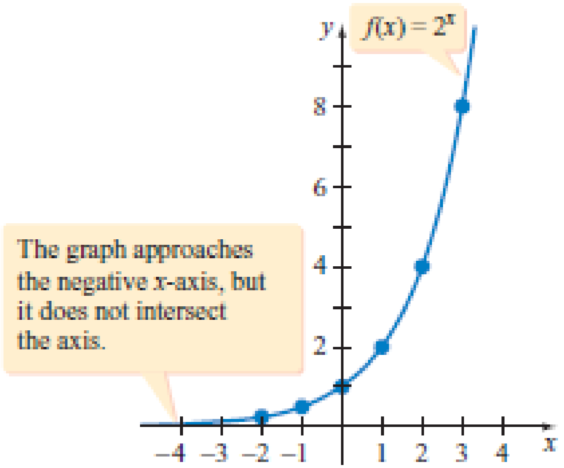

Figure 4.15

Table 4.2

x	$y = f(x) = 2^x$	(x, y)
-2	$f(-2) = 2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$f(0) = 2^0 = 1$	$(0, 1)$
1	$f(1) = 2^1 = 2$	$(1, 2)$
2	$f(2) = 2^2 = 4$	$(2, 4)$
3	$f(3) = 2^3 = 8$	$(3, 8)$

Note the following properties of the graph of the exponential function $f(x) = 2^x$.

- The y-intercept is $(0, 1)$.
- The graph passes through $(1, 2)$.
- As x decreases without bound (that is, as $x \to -\infty$), $f(x) \to 0$.
- The graph is a smooth, continuous increasing curve.

EXAMPLE 2 Graph an Exponential Function

Graph: $g(x) = \left(\frac{3}{4}\right)^x$

Solution

Because the base $\frac{3}{4}$ is less than 1, we know that the graph of g is a decreasing function that is asymptotic to the positive x-axis. The y-intercept of the graph is the point (0, 1),

(continued)

and the graph passes through $\left(1, \frac{3}{4}\right)$. Plot a few additional points (see Table 4.4), and then draw a smooth curve through the points, as in Figure 4.18.

Table 4.4

x	$y = g(x) = \left(\frac{3}{4}\right)^x$	(x, y)
−3	$\left(\frac{3}{4}\right)^{-3} = \frac{64}{27}$	$\left(-3, \frac{64}{27}\right)$
−2	$\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}$	$\left(-2, \frac{16}{9}\right)$
−1	$\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$	$\left(-1, \frac{4}{3}\right)$
2	$\left(\frac{3}{4}\right)^{2} = \frac{9}{16}$	$\left(2, \frac{9}{16}\right)$
3	$\left(\frac{3}{4}\right)^{3} = \frac{27}{64}$	$\left(3, \frac{27}{64}\right)$

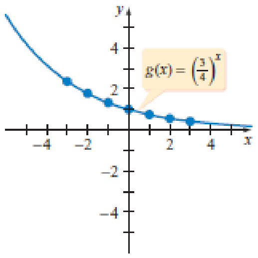

Figure 4.18

Natural Exponential Function

The irrational number π is often used in applications that involve circles. Another irrational number, denoted by the letter e, is useful in many applications that involve growth or decay.

Definition of e

The letter e represents the number that

$$\left(1 + \frac{1}{n}\right)^n$$

approaches as n increases without bound.

The letter e was chosen in honor of the Swiss mathematician Leonhard Euler. He was able to compute the value of e to several decimal places by evaluating $\left(1 + \frac{1}{n}\right)^n$ for large values of n, as shown in Table 4.5 on page 352.

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Table 4.5

Value of n	Value of $\left(1 + \frac{1}{n}\right)^n$
1	2
10	2.59374246
100	2.704813829
1000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281693

The value of e accurate to eight decimal places is 2.71828183.

The base of an exponential function can be any positive real number other than 1. The number 10 is a convenient base to use for some situations, but we will see that the number e is often the best base to use in real-life applications. The exponential function with e as the base is known as the *natural exponential function*.

Definition of the Natural Exponential Function

For all real numbers x, the function defined by

$$f(x) = e^x$$

is called the **natural exponential function.**

Logarithmic Functions

Every exponential function of the form $g(x) = b^x$ is a one-to-one function and therefore has an inverse function. Sometimes we can determine the inverse of a function represented by an equation by interchanging the variables of its equation and then solving for the dependent variable. If we attempt to use this procedure for $g(x) = b^x$, we obtain

$$g(x) = b^x$$
$$y = b^x$$
$$x = b^y \qquad \text{• Interchange the variables.}$$

None of our previous methods can be used to solve the equation $x = b^y$ for the exponent y. Thus we need to develop a new procedure. One method would be to merely write

$$y = \text{the power of } b \text{ that produces } x$$

Although this would work, it is not concise. We need a compact notation to represent "y is the power of b that produces x." This more compact notation is given in the following definition.

Math Matters

Logarithms were developed by John Napier (1550–1617) as a means of simplifying the calculations of astronomers. One of his ideas was to devise a method by which the product of two numbers could be determined by performing an addition.

Definition of a Logarithm and a Logarithmic Function

If $x > 0$ and b is a positive constant $(b \neq 1)$, then

$$y = \log_b x \qquad \text{if and only if} \qquad b^y = x$$

The notation $\log_b x$ is read "the **logarithm** (or log) base b of x." The function defined by $f(x) = \log_b x$ is a **logarithmic function** with base b. This function is the inverse of the exponential function $g(x) = b^x$.

It is essential to remember that $f(x) = \log_b x$ is the inverse function of $g(x) = b^x$. Because these functions are inverses and because functions that are inverses have the property that $f(g(x)) = x$ and $g(f(x)) = x$, we have the following important relationships.

Composition of Logarithmic and Exponential Functions

Let $g(x) = b^x$ and $f(x) = \log_b x$ $(x > 0, b > 0, b \neq 1)$. Then

$$g(f(x)) = b^{\log_b x} = x \qquad \text{and} \qquad f(g(x)) = \log_b b^x = x$$

As an example of these relationships, let $g(x) = 2^x$ and $f(x) = \log_2 x$. Then

$$2^{\log_2 x} = x \qquad \text{and} \qquad \log_2 2^x = x$$

The equations

$$y = \log_b x \qquad \text{and} \qquad b^y = x$$

are different ways of expressing the same concept.

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Definition of Exponential Form and Logarithmic Form

The **exponential form** of $y = \log_b x$ is $b^y = x$.

The **logarithmic form** of $b^y = x$ is $y = \log_b x$.

EXAMPLE 1 Change from Logarithmic to Exponential Form

Write each equation in its exponential form.

a. $3 = \log_2 8$ **b.** $2 = \log_{10}(x + 5)$ **c.** $\log_e x = 4$ **d.** $\log_b b^3 = 3$

Solution

Use the definition $y = \log_b x$ if and only if $b^y = x$.

Logarithms are exponents.

a. $3 = \log_2 8$ if and only if $2^3 = 8$

Base

(continued)

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b. $2 = \log_{10}(x + 5)$ if and only if $10^2 = x + 5$.

c. $\log_e x = 4$ if and only if $e^4 = x$.

d. $\log_b b^3 = 3$ if and only if $b^3 = b^3$.

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EXAMPLE 2 Change from Exponential to Logarithmic Form

Write each equation in its logarithmic form.

a. $3^2 = 9$ **b.** $5^3 = x$

c. $a^b = c$ **d.** $b^{\log_b 5} = 5$

Solution

The logarithmic form of $b^y = x$ is $y = \log_b x$.

Exponent

a. $3^2 = 9$ if and only if $2 = \log_3 9$

Base

b. $5^3 = x$ if and only if $3 = \log_5 x$.

c. $a^b = c$ if and only if $b = \log_a c$.

d. $b^{\log_b 5} = 5$ if and only if $\log_b 5 = \log_b 5$.

■ **Try Exercise 14, page 366**

The definition of a logarithm and the definition of an inverse function can be used to establish many properties of logarithms. For instance,

- $\log_b b = 1$ because $b = b^1$.
- $\log_b 1 = 0$ because $1 = b^0$.
- $\log_b(b^x) = x$ because $b^x = b^x$.
- $b^{\log_b x} = x$ because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions. Thus $g[f(x)] = x$.

We will refer to the preceding properties as the *basic logarithmic properties*.

Basic Logarithmic Properties

1. $\log_b b = 1$ **2.** $\log_b 1 = 0$ **3.** $\log_b(b^x) = x$ **4.** $b^{\log_b x} = x$

EXAMPLE 3 Apply the Basic Logarithmic Properties

Evaluate each of the following logarithms.

a. $\log_8 1$ **b.** $\log_5 5$

c. $\log_2(2^4)$ **d.** $3^{\log_3 7}$

Solution

a. By property 2, $\log_8 1 = 0$.

b. By property 1, $\log_5 5 = 1$.

c. By property 3, $\log_2(2^4) = 4$.

d. By property 4, $3^{\log_3 7} = 7$.

■ **Try Exercise 32, page 366**

Some logarithms can be evaluated just by remembering that a logarithm is an exponent. For instance, $\log_5 25$ equals 2 because the base 5 raised to the second power equals 25.

- $\log_{10} 100 = 2$ because $10^2 = 100$.
- $\log_4 64 = 3$ because $4^3 = 64$.
- $\log_7 \frac{1}{49} = -2$ because $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$.

Common and Natural Logarithms

Two of the most frequently used logarithmic functions are *common logarithms,* which have base 10, and *natural logarithms,* which have base e (the base of the natural exponential function).

Definition of Common and Natural Logarithms

The function defined by $f(x) = \log_{10} x$ is called the **common logarithmic function.** It is customarily written as $f(x) = \log x$, without stating the base.

The function defined by $f(x) = \log_e x$ is called the **natural logarithmic function.** It is customarily written as $f(x) = \ln x$.

Most scientific or graphing calculators have a LOG key for evaluating common logarithms and an LN key to evaluate natural logarithms. For instance, using a graphing calculator,

$$\log 24 \approx 1.3802112 \quad \text{and} \quad \ln 81 \approx 4.3944492$$

The graphs of $f(x) = \log x$ and $f(x) = \ln x$ can be drawn using the same techniques we used to draw the graphs in the preceding examples. However, these graphs also can be produced

Properties of Logarithms

In the following properties, b, M, and N are positive real numbers ($b \neq 1$).

Product property	$\log_b(MN) = \log_b M + \log_b N$
Quotient property	$\log_b \frac{M}{N} = \log_b M - \log_b N$
Power property	$\log_b(M^p) = p \log_b M$
Logarithm-of-each-side property	$M = N$ implies $\log_b M = \log_b N$
One-to-one property	$\log_b M = \log_b N$ implies $M = N$

Here is a proof of the product property.

Proof

Let $r = \log_b M$ and $s = \log_b N$. These equations can be written in exponential form as

$$M = b^r \quad \text{and} \quad N = b^s$$

Now consider the product MN.

$$MN = b^r b^s \quad \text{• Substitute for } M \text{ and } N.$$

$$MN = b^{r+s} \quad \text{• Product property of exponents}$$

$$\log_b MN = r + s \quad \text{• Write in logarithmic form.}$$

$$\log_b MN = \log_b M + \log_b N \quad \text{• Substitute for } r \text{ and } s.$$

The last equation is our desired result. ◆

The quotient property and the power property can be proved in a similar manner. See Exercises 87 and 88 on page 380.

The properties of logarithms are often used to rewrite logarithmic expressions in an equivalent form. The process of using the product or quotient rules to rewrite a single logarithm as the sum or difference of two or more logarithms, or using the power property to rewrite $\log_b(M^p)$ in its equivalent form $p \log_b M$, is called **expanding the logarithmic expression**. We illustrate this process in Example 1.

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EXAMPLE 1 Expand Logarithmic Expressions

Use the properties of logarithms to expand the following logarithmic expressions. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions.

a. $\log_5(xy^2)$ **b.** $\ln\left(\frac{e\sqrt{y}}{z^3}\right)$

Solution

a.
$$\begin{aligned}\log_5(xy^2) &= \log_5 x + \log_5 y^2 && \text{• Product property}\\ &= \log_5 x + 2\log_5 y && \text{• Power property}\end{aligned}$$

b.
$$\begin{aligned}\ln\left(\frac{e\sqrt{y}}{z^3}\right) &= \ln(e\sqrt{y}) - \ln z^3 && \text{• Quotient property}\\ &= \ln e + \ln\sqrt{y} - \ln z^3 && \text{• Product property}\\ &= \ln e + \ln y^{1/2} - \ln z^3 && \text{• Write } \sqrt{y} \text{ as } y^{1/2}.\\ &= \ln e + \frac{1}{2}\ln y - 3\ln z && \text{• Power property}\\ &= 1 + \frac{1}{2}\ln y - 3\ln z && \text{• Evaluate } \ln e.\end{aligned}$$

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EXAMPLE 2 Condense Logarithmic Expressions

Use the properties of logarithms to rewrite each expression as a single logarithm with a coefficient of 1. Assume all variable expressions represent positive real numbers.

a. $2 \ln x + \frac{1}{2}\ln(x+4)$ **b.** $\log_5(x^2-4) + 3\log_5 y - \log_5(x-2)^2$

Solution

a. $2 \ln x + \frac{1}{2}\ln(x+4) = \ln x^2 + \ln(x+4)^{1/2}$ • Power property

$= \ln[x^2(x+4)^{1/2}]$ • Product property

$= \ln[x^2\sqrt{(x+4)}]$ • Rewriting $(x+4)^{1/2}$ as $\sqrt{x+4}$ is an optional step.

b. $\log_5(x^2-4) + 3\log_5 y - \log_5(x-2)^2$

$= \log_5(x^2-4) + \log_5 y^3 - \log_5(x-2)^2$ • Power property

$= [\log_5(x^2-4) + \log_5 y^3] - \log_5(x-2)^2$ • Order of Operations Agreement

$= \log_5[(x^2-4)y^3] - \log_5(x-2)^2$ • Product property

$= \log_5\left[\frac{(x^2-4)y^3}{(x-2)^2}\right]$ • Quotient property

$= \log_5\left[\frac{(x+2)(x-2)y^3}{(x-2)^2}\right]$ • Factor.

$= \log_5\left[\frac{(x+2)y^3}{x-2}\right]$ • Simplify.

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 Eight Edition,
- Blitzer, Robert , Precalculus , 5th edition