ALGEBRA

PREPARED BY ENGR. JOHN R. REJANO,ECE

LEARNING OUTCOMES

- ◆ Solve for linear and quadratic equations
- ◆ Solve word problems involving linear and quadratic equations

ALGEBRA

Linear Equations

- An **equation** is a statement about the equality of two expressions. If either of the expressions contains a variable, the equation may be a true statement for some values of the variable and a false statement for other values.

- For example, the equation 2x + 1 = 7 is a true statement for x=3 but it is false for any number except 3. The number 3 is said to **satisfy** the equation 2x +1 = 7 because substituting 3 for x
- 2(3) + 1 = 7, produces which is a true statement.

- To **solve** an equation means to find all values of the variable that satisfy the equation.
- The values that satisfy an equation are called **solutions** or **roots** of the equation.

  For instance, 2 is a solution of $x + 3 = 5$.

Equivalent equations are equations that have exactly the same solution or solutions. The process of solving an equation is often accomplished by producing a sequence of equivalent equations until we arrive at an equation or equations of the form

Variable = Constant

ALGEBRA

Addition and Subtraction Property of Equality

Adding the same expression to each side of an equation or subtracting the same expression from each side of an equation produces an equivalent equation.

EXAMPLE

Begin with the equation $2x - 7 = 11$. Replacing x with 9 shows that 9 is a solution of the equation. Now add 7 to each side of the equation. The resulting equation is $2x = 18$, and the solution of the new equation is still 9.

Multiplication and Division Property of Equality

Multiplying or dividing each side of an equation by the same nonzero expression produces an equivalent equation.

EXAMPLE

Begin with the equation $\frac{2}{3}x = 8$. Replacing x with 12 shows that 12 is a solution of the equation. Now multiply each side of the equation by $\frac{3}{2}$. The resulting equation is $x = 12$, and the solution of the new equation is still 12.

Definition of a Linear Equation

A **linear equation,** or first-degree equation, in the single variable x is an equation that can be written in the form

$$ax + b = 0$$

where a and b are real numbers, with $a \neq 0$.

Linear equations are solved by applying the properties of real numbers and the properties of equality.

ALGEBRA

EXAMPLE 1 Solve a Linear Equation in One Variable

Solve: $3x - 5 = 7x - 11$

ALGEBRA

EXAMPLE 1 Solve a Linear Equation in One Variable

Solve: $3x - 5 = 7x - 11$

Solution

$$3x - 5 = 7x - 11$$

$$3x - 7x - 5 = 7x - 7x - 11$$

- Subtract $7x$ from each side of the equation.

$$-4x - 5 = -11$$

$$-4x - 5 + 5 = -11 + 5$$

- Add 5 to each side of the equation.

$$-4x = -6$$

$$\frac{-4x}{-4} = \frac{-6}{-4}$$

- Divide each side of the equation by -4.

$$x = \frac{3}{2}$$

- The equation is now in the form Variable = Constant.

As shown to the left, $\frac{3}{2}$ satisfies the original equation. The solution is $\frac{3}{2}$.

ALGEBRA

EXAMPLE 2 Solve a Linear Equation in One Variable

Solve: $8 - 5(2x - 7) = 3(16 - 5x) + 5$

ALGEBRA

Solution

$$8 - 5(2x - 7) = 3(16 - 5x) + 5$$

$$8 - 10x + 35 = 48 - 15x + 5$$ • Use the distributive property.

$$-10x + 43 = -15x + 53$$ • Simplify.

$$-10x + 15x + 43 = -15x + 15x + 53$$ • Add $15x$ to each side of the equation.

$$5x + 43 = 53$$

$$5x + 43 - 43 = 53 - 43$$ • Subtract 43 from each side of the equation.

$$5x = 10$$

$$\frac{5x}{5} = \frac{10}{5}$$ • Divide each side of the equation by 5.

$$x = 2$$ • Check in the original equation.

The solution is 2.

ALGEBRA

OTHER EXAMPLES

- **8.** $6 - 2(4x + 1) = 3x - 2(2x + 5)$

8. $6 - 2(4x + 1) = 3x - 2(2x + 5)$

$$
\begin{aligned}
6 - 8x - 2 &= 3x - 4x - 10 \\
-8x + 4 &= -x - 10 \\
-8x + x + 4 &= -x + x - 10 \\
-7x + 4 &= -10 \\
-7x + 4 - 4 &= -10 - 4 \\
-7x &= -14 \\
\frac{-7x}{-7} &= \frac{-14}{-7} \\
x &= 2
\end{aligned}
$$

EXAMPLE 3 Solve by Clearing Fractions

Solve: $\frac{2}{3}x + 10 - \frac{x}{5} = \frac{36}{5}$

Solution

$$\frac{2}{3}x + 10 - \frac{x}{5} = \frac{36}{5}$$

$$15\left(\frac{2}{3}x + 10 - \frac{x}{5}\right) = 15\left(\frac{36}{5}\right)$$

- Multiply each side of the equation by 15, the LCD of all denominators.

$$10x + 150 - 3x = 108$$

- Simplify.

$$7x + 150 = 108$$

$$7x + 150 - 150 = 108 - 150$$

- Subtract 150 from each side.

$$7x = -42$$

$$\frac{7x}{7} = \frac{-42}{7}$$

- Divide each side by 7.

$$x = -6$$

- Check in the original equation.

The solution is -6.

ALGEBRA

OTHER EXAMPLES

■ **14.** $\frac{1}{2}x + 7 - \frac{1}{4}x = \frac{19}{2}$

$x = 2$

14.

$$\frac{1}{2}x + 7 - \frac{1}{4}x = \frac{19}{2}$$

$$4\left(\frac{1}{2}x + 7 - \frac{1}{4}x\right) = 4\left(\frac{19}{2}\right)$$ • Multiply each side by 4.

$$2x + 28 - x = 38$$

$$x = 38 - 28$$ • Collect like terms.

$$x = 10$$

Contradictions, Conditional Equations, and Identities

An equation that has no solutions is called a **contradiction.** The equation $x = x + 1$ is a contradiction. No number is equal to itself increased by 1.

An equation that is true for some values of the variable but not true for other values of the variable is called a **conditional equation.** For example, $x + 2 = 8$ is a conditional equation because it is true for $x = 6$ and false for any number not equal to 6.

An **identity** is an equation that is true for all values of the variable for which all terms of the equation are defined. Examples of identities include the equations $x + x = 2x$ and $4(x + 3) - 1 = 4x + 11$.

EXAMPLE 4 Classify Equations

Classify each equation as a contradiction, a conditional equation, or an identity.

a. $x + 1 = x + 4$

b. $4x + 3 = x - 9$

c. $5(3x - 2) - 7(x - 4) = 8x + 18$

Solution

a. Subtract x from both sides of $x + 1 = x + 4$ to produce the equivalent equation $1 = 4$. Because $1 = 4$ is a false statement, the original equation $x + 1 = x + 4$ has no solutions. It is a contradiction.

b. Solve using the procedures that produce equivalent equations.

$$4x + 3 = x - 9$$

$$3x + 3 = -9 \quad \text{• Subtract } x \text{ from each side.}$$

$$3x = -12 \quad \text{• Subtract 3 from each side.}$$

$$x = -4 \quad \text{• Divide each side by 3.}$$

Check to confirm that -4 is a solution. The equation $4x + 3 = x - 9$ is true for $x = -4$, but it is not true for any other values of x. Thus $4x + 3 = x - 9$ is a conditional equation.

c. Simplify the left side of the equation to show that it is *identical* to the right side.

$$5(3x - 2) - 7(x - 4) = 8x + 18$$

$$15x - 10 - 7x + 28 = 8x + 18$$

$$8x + 18 = 8x + 18$$

The original equation $5(3x - 2) - 7(x - 4) = 8x + 18$ is true for all real numbers x. The equation is an identity.

Absolute Value Equations

The absolute value of a real number x is the distance between the number x and the number 0 on the real number line. Thus the solutions of $|x| = 3$ are all real numbers that are 3 units from 0. Therefore, the solutions of $|x| = 3$ are $x = 3$ or $x = -3$. See Figure 1.1.

The following property is used to solve absolute value equations.

A Property of Absolute Value Equations

For any variable expression E and any nonnegative real number k,

$$|E| = k \quad \text{if and only if} \quad E = k \quad \text{or} \quad E = -k$$

EXAMPLE

If $|x| = 5$, then $x = 5$ or $x = -5$.

If $|x| = \frac{3}{2}$, then $x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

If $|x| = 0$, then $x = 0$.

EXAMPLE 5 Solve an Absolute Value Equation

Solve: $|2x - 5| = 21$

Solution

$|2x - 5| = 21$ implies $2x - 5 = 21$ or $2x - 5 = -21$. Solving each of these linear equations produces

$$\begin{aligned} 2x - 5 &= 21 \quad &\text{or} \quad 2x - 5 &= -21 \\ 2x &= 26 & 2x &= -16 \\ x &= 13 & x &= -8 \end{aligned}$$

The solutions are -8 and 13.

ALGEBRA

ASSIGNMENT TASK 1

SOLVE

- 38. $|2x - 3| = 21$

Applications of Linear Equations

Linear equations often can be used to model real-world data.

EXAMPLE 6 Movie Theater Ticket Prices

Movie theater ticket prices have been increasing steadily in recent years (see Table 1.1). An equation that models the average U.S. movie theater ticket price p, in dollars, is given by

$$p = 0.211t + 5.998$$

where t is the number of years after 2003. (This means that $t = 0$ corresponds to 2003.) Use this equation to predict the year in which the average U.S. movie theater ticket price will reach $7.50.

Solution

$$p = 0.211t + 5.998$$

$$7.50 = 0.211t + 5.998 \quad \text{• Substitute 7.50 for } p.$$

$$1.502 = 0.211t \quad \text{• Solve for } t.$$

$$t \approx 7.1$$

Our equation predicts that the average U.S. movie theater ticket price will reach $7.50 about 7.1 years after 2003, which is 2010.

Table 1.1 Average U.S. Movie Theater Ticket Price

Year	Price (in dollars)
2003	6.03
2004	6.21
2005	6.41
2006	6.55
2007	6.88
2008	7.08

Source: National Association of Theatre Owners, http:/www.natoonline.org/statisticstickets.htm.

ALGEBRA

■ **50.** **Health** According to one formula for lean body mass (LBM, in kilograms) given by R. Hume, the mass of the body minus fat is

$$\text{LBM} = 0.3281W + 0.3393H - 29.5336$$

where W is a person's weight in kilograms and H is the person's height in centimeters. If a person is 175 centimeters tall, what should that person weigh to have an LBM of 55 kilograms? Round to the nearest kilogram.

50. Replace LBM by 55 and H by 175 in the equation $\text{LBM} = 0.3281W + 0.3393H - 29.5336$, and then solve for W.

$$\begin{aligned} \text{LBM} &= 0.3281W + 0.3393H - 29.5336 \\ 55 &= 0.3281W + 0.3393(175) - 29.5336 \\ 55 &= 0.3281W + 29.8439 \\ 25.1561 &= 0.3281W \\ 77 &\approx W \end{aligned}$$

The person should weigh approximately 77 kilograms.

EXAMPLE 7 Driving Time

Alicia is driving along a highway that passes through Centerville (see Figure 1.2). Her distance d, in miles, from Centerville is given by the equation

$$d = |135 - 60t|$$

where t is the time in hours since the start of her trip and $0 \le t \le 5$. Determine when Alicia will be exactly 15 miles from Centerville.

Solution

Substitute 15 for d.

$$d = |135 - 60t|$$

$$15 = |135 - 60t|$$

$$15 = 135 - 60t \quad \text{or} \quad -15 = 135 - 60t \qquad \text{• Solve for } t.$$

$$-120 = -60t \qquad\qquad -150 = -60t$$

$$2 = t \qquad\qquad \frac{5}{2} = t$$

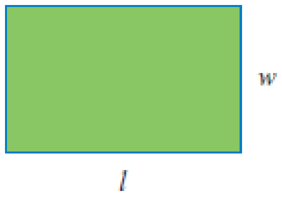
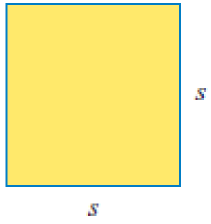
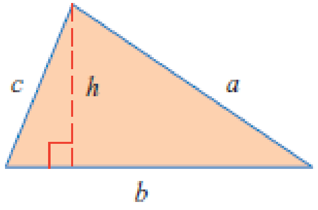
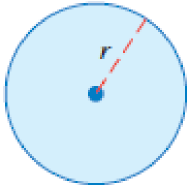
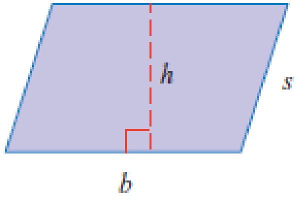
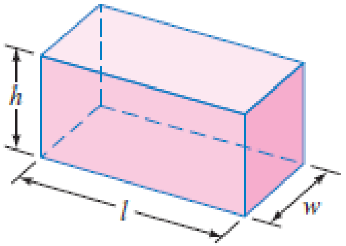
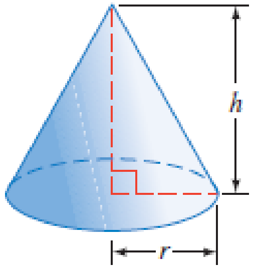
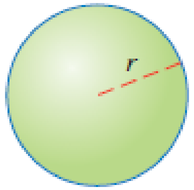
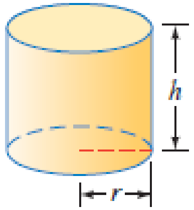
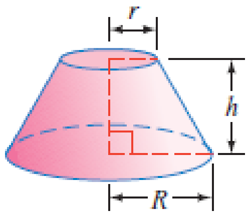
Alicia will be exactly 15 miles from Centerville after she has driven for 2 hours and after she has driven for $2\frac{1}{2}$ hours.

Formulas

A **formula** is an equation that expresses known relationships between two or more variables. Table 1.2 lists several formulas from geometry that are used in this text. The variable P represents perimeter, C represents circumference of a circle, A represents area, S represents surface area of an enclosed solid, and V represents volume.

INTERVENTION FOR CALCULUS

Table 1.2 Formulas from Geometry

Rectangle	Square	Triangle	Circle	Parallelogram
$P = 2l + 2w$ $A = lw$	$P = 4s$ $A = s^2$	$P = a + b + c$ $A = \frac{1}{2}bh$	$C = \pi d = 2\pi r$ $A = \pi r^2$	$P = 2b + 2s$ $A = bh$
				
Rectangular Solid	**Right Circular Cone**	**Sphere**	**Right Circular Cylinder**	**Frustum of a Cone**
$S = 2(wh + lw + hl)$ $V = lwh$	$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$ $V = \frac{1}{3}\pi r^2 h$	$S = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$	$S = 2\pi rh + 2\pi r^2$ $V = \pi r^2 h$	$S = \pi(R + r)\sqrt{h^2 + (R - r)^2} + \pi r^2 + \pi R^2$ $V = \frac{1}{3}\pi h(r^2 + rR + R^2)$
				

INTERVENTION FOR CALCULUS

EXAMPLE 1 Solve a Formula for a Specified Variable

a. Solve $2l + 2w = P$ for l.

b. Solve $S = 2(wh + lw + hl)$ for h.

Solution

a. $2l + 2w = P$

$$2l = P - 2w$$
• Subtract $2w$ from each side to isolate the $2l$ term.

$$l = \frac{P - 2w}{2}$$
• Divide each side by 2.

b. $S = 2(wh + lw + hl)$

$$S = 2wh + 2lw + 2hl$$

$$S - 2lw = 2wh + 2hl$$
• Isolate the terms that involve the variable h on the right side.

$$S - 2lw = 2h(w + l)$$
• Factor $2h$ from the right side.

$$\frac{S - 2lw}{2(w + l)} = h$$
• Divide each side by $2(w + l)$.

INTERVENTION FOR CALCULUS

Applications

Linear equations emerge in a variety of application problems. In solving such problems, it generally helps to apply specific techniques in a series of small steps. The following general strategies should prove helpful in the remaining portion of this section.

Strategies for Solving Application Problems

1. Read the problem carefully. If necessary, reread the problem several times.
2. When appropriate, draw a sketch and label parts of the drawing with the specific information given in the problem.
3. Determine the unknown quantities, and label them with variables. Write down any equation that relates the variables.
4. Use the information from step 3, along with a known formula or some additional information given in the problem, to write an equation.
5. Solve the equation obtained in step 4, and check to see whether the results satisfy all the conditions of the original problem.

INTERVENTION FOR CALCULUS

EXAMPLE 3 Dimensions of a Painting

One of the best known paintings is the *Mona Lisa* by Leonardo da Vinci. It is on display at the Musée du Louvre, in Paris. The length (or height) of this rectangular-shaped painting is 24 centimeters more than its width. The perimeter of the painting is 260 centimeters. Find the width and length of the painting.

Gianni Dagli Orti/CORBIS

Solution

1. Read the problem carefully.
2. Draw a rectangle. See Figure 1.3.
3. Label the rectangle. We have used w for its width and l for its length. The problem states that the length is 24 centimeters more than the width. Thus l and w are related by the equation

$$l = w + 24$$

4. The perimeter of a rectangle is given by the formula $P = 2l + 2w$. To produce an equation that involves only constants and a single variable (say, w), substitute 260 for P and $w + 24$ for l.

$$P = 2l + 2w$$
$$260 = 2(w + 24) + 2w$$

5. Solve for w.

$$260 = 2w + 48 + 2w$$
$$260 = 4w + 48$$
$$212 = 4w$$
$$w = 53$$

The length is 24 centimeters more than the width. Thus $l = 53 + 24 = 77$.

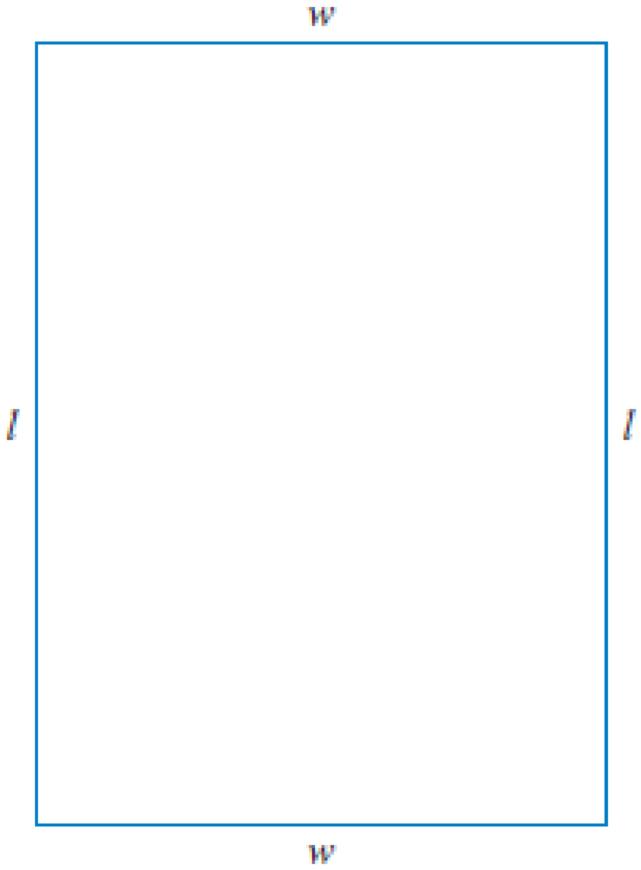
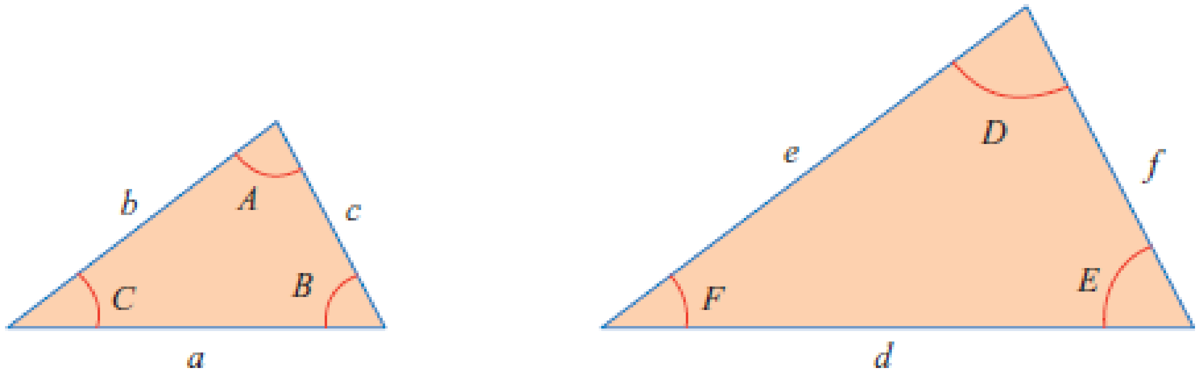


Figure 1.3

INTERVENTION FOR CALCULUS

Similar triangles are ones for which the measures of corresponding angles are equal. The triangles below are similar.



$\angle A = \angle D \qquad \angle B = \angle E \qquad \angle C = \angle F$

An important relationship among the sides of similar triangles is that the ratios of corresponding sides are equal. Thus, for the triangles above,

$$\frac{a}{b} = \frac{d}{e} \qquad \frac{a}{c} = \frac{d}{f} \qquad \frac{b}{c} = \frac{e}{f}$$

This fact is used in many applications.

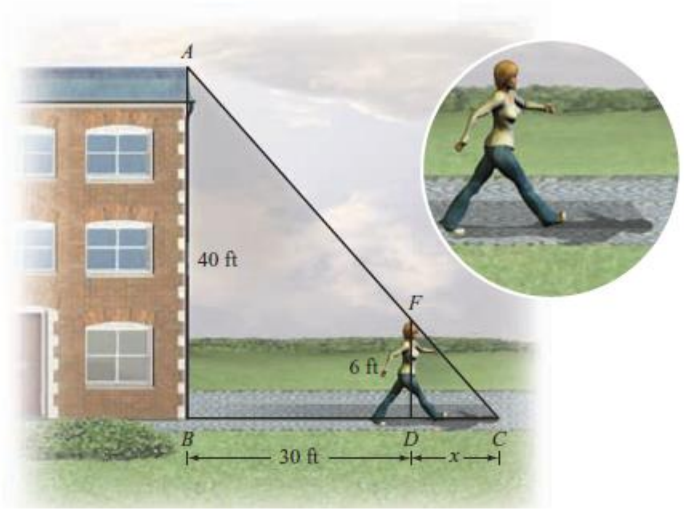
EXAMPLE 4 A Problem Involving Similar Triangles

A person 6 feet tall is in the shadow of a building 40 feet tall and is walking directly away from the building. When the person is 30 feet from the building, the tip of the person's shadow is at the same point as the tip of the shadow of the building. How much farther must the person walk to be just out of the shadow of the building? Round to the nearest tenth of a foot.

INTERVENTION FOR CALCULUS

Solution

Let x be the distance the person has to walk. Draw a picture of the situation using similar triangles.



Triangles ABC and FDC are similar triangles. Therefore, the ratios of the lengths of the corresponding sides are equal. Using this fact, we can write an equation.

$$\frac{30 + x}{40} = \frac{x}{6}$$

Now solve the equation.

$$\frac{30 + x}{40} = \frac{x}{6}$$

$$120\left(\frac{30 + x}{40}\right) = 120\left(\frac{x}{6}\right)$$ • Multiply each side by 120, the LCD of 40 and 6.

$$3(30 + x) = 20x$$ • Solve for x.

$$90 + 3x = 20x$$

$$90 = 17x$$

$$5.3 \approx x$$

The person must walk an additional 5.3 feet.

INTERVENTION FOR CALCULUS

Percent mixture problems involve combining solutions or alloys that have different concentrations of a common substance. Percent mixture problems can be solved by using the formula $pA = Q$, where p is the percent of concentration (in decimal form), A is the amount of the solution or alloy, and Q is the quantity of a substance in the solution or alloy. For example, in 4 liters of a 25% acid solution, p is the percent of acid (0.25 as a decimal), A is the amount of solution (4 liters), and Q is the amount of acid in the solution, which equals $(0.25)(4)$ liters $= 1$ liter.

EXAMPLE 8 A Percent Mixture Problem

A chemist mixes an 11% hydrochloric acid solution with a 6% hydrochloric acid solution. How many milliliters of each solution should the chemist use to make a 600-milliliter solution that is 8% hydrochloric acid?

Solution

Let x be the number of milliliters of the 11% solution. Because the solution after mixing will have a total of 600 milliliters of fluid, $600 - x$ is the number of milliliters of the 6% solution. See Figure 1.5.

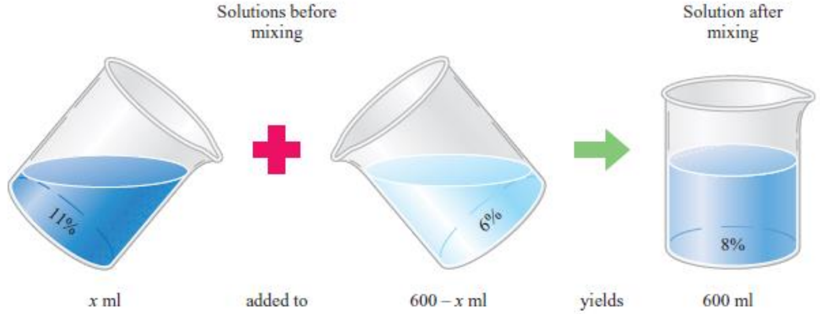


Figure 1.5

Because all the hydrochloric acid in the solution after mixing comes from either the 11% solution or the 6% solution, the number of milliliters of hydrochloric acid in the 11% solution added to the number of milliliters of hydrochloric acid in the 6% solution must equal the number of milliliters of hydrochloric acid in the 8% solution.

$$\begin{pmatrix}\text{ml of acid in}\\ \text{11\% solution}\end{pmatrix} + \begin{pmatrix}\text{ml of acid in}\\ \text{6\% solution}\end{pmatrix} = \begin{pmatrix}\text{ml of acid in}\\ \text{8\% solution}\end{pmatrix}$$

$$0.11x + 0.06(600 - x) = 0.08(600)$$

$$0.11x + 36 - 0.06x = 48$$

$$0.05x + 36 = 48$$

$$0.05x = 12$$

$$x = 240$$

The chemist should use 240 milliliters of the 11% solution and 360 milliliters of the 6% solution to make a 600-milliliter solution that is 8% hydrochloric acid.

INTERVENTION FOR CALCULUS

EXAMPLE 10 A Work Problem

Pump A can fill a pool in 6 hours, and pump B can fill the same pool in 3 hours. How long will it take to fill the pool if both pumps are used?

Solution

Because pump A fills the pool in 6 hours, $\frac{1}{6}$ represents the part of the pool filled by pump A in 1 hour. Because pump B fills the pool in 3 hours, $\frac{1}{3}$ represents the part of the pool filled by pump B in 1 hour.

Let t equal the number of hours to fill the pool using both pumps. Then

$$t \cdot \frac{1}{6} = \frac{t}{6}$$ • Part of the pool filled by pump A

$$t \cdot \frac{1}{3} = \frac{t}{3}$$ • Part of the pool filled by pump B

$$\begin{pmatrix}\text{Part filled}\\ \text{by pump A}\end{pmatrix} + \begin{pmatrix}\text{Part filled}\\ \text{by pump B}\end{pmatrix} = \begin{pmatrix}\text{1 filled}\\ \text{pool}\end{pmatrix}$$

$$\frac{t}{6} + \frac{t}{3} = 1$$

Multiplying each side of the equation by 6 produces

$$t + 2t = 6$$
$$3t = 6$$
$$t = 2$$

Check: Pump A fills $\frac{2}{6}$, or $\frac{1}{3}$, of the pool in 2 hours and pump B fills $\frac{2}{3}$ of the pool in 2 hours, so 2 hours is the time required to fill the pool if both pumps are used.

ASSIGNMENT TASK 2

Solve the problem

■ 56. **Print a Report** Printer A can print a report in 3 hours. Printer B can print the same report in 4 hours. How long would it take both printers, working together, to print the report?

QUADRATIC EQUATIONS

Quadratic Equations

Solving Quadratic Equations by Factoring

In Section 1.1 you solved linear equations. In this section you will learn to solve a type of equation that is referred to as a *quadratic equation*.

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the **standard quadratic form**

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

Several methods can be used to solve a quadratic equation. For instance, if you can factor $ax^2 + bx + c$ into linear factors, then $ax^2 + bx + c = 0$ can be solved by applying the following property.

The Zero Product Principle

If A and B are algebraic expressions such that $AB = 0$, then $A = 0$ or $B = 0$.

The zero product principle states that if the product of two factors is zero, then at least one of the factors must be zero.

INTERVENTION FOR CALCULUS

EXAMPLE 1 Solve by Factoring

Solve each quadratic equation by factoring.

a. $x^2 + 2x - 15 = 0$ **b.** $2x^2 - 5x = 12$

Solution

a.

$$x^2 + 2x - 15 = 0$$

$$(x - 3)(x + 5) = 0 \qquad \text{• Factor.}$$

$$x - 3 = 0 \quad \text{or} \quad x + 5 = 0 \qquad \text{• Set each factor equal to zero.}$$

$$x = 3 \qquad\qquad x = -5 \qquad \text{• Solve each linear equation.}$$

A check shows that 3 and -5 are both solutions of $x^2 + 2x - 15 = 0$.

b.

$$2x^2 - 5x = 12$$

$$2x^2 - 5x - 12 = 0 \qquad \text{• Write in standard quadratic form.}$$

$$(x - 4)(2x + 3) = 0 \qquad \text{• Factor.}$$

$$x - 4 = 0 \quad \text{or} \quad 2x + 3 = 0 \qquad \text{• Set each factor equal to zero.}$$

$$x = 4 \qquad\qquad 2x = -3 \qquad \text{• Solve each linear equation.}$$

$$x = -\frac{3}{2}$$

A check shows that 4 and $-\frac{3}{2}$ are both solutions of $2x^2 - 5x = 12$.

INTERVENTION FOR CALCULUS

Some quadratic equations have a solution that is called a *double root.* For instance, consider $x^2 - 8x + 16 = 0$. Solving this equation by factoring, we have

$$x^2 - 8x + 16 = 0$$

$$(x - 4)(x - 4) = 0 \qquad \text{• Factor.}$$

$$x - 4 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{• Set each factor equal to zero.}$$

$$x = 4 \qquad\qquad x = 4 \qquad \text{• Solve each linear equation.}$$

The only solution of $x^2 - 8x + 16 = 0$ is 4. In this situation, the single solution 4 is called a **double solution** or **double root** because it was produced by solving the two identical equations $x - 4 = 0$, both of which have 4 as a solution.

INTERVENTION FOR CALCULUS

ASSIGNMENT TASK 3

Solve the quadratic equation by factoring

■6. $12x^2 - 41x + 24 = 0$

Solving Quadratic Equations by Taking Square Roots

Recall that $\sqrt{x^2} = |x|$. This principle can be used to solve some quadratic equations by taking the square root of each side of the equation.

In the following example, we use this idea to solve $x^2 = 25$.

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$ • Take the square root of each side.

$$|x| = 5$$ • Use the fact that $\sqrt{x^2} = |x|$ and $\sqrt{25} = 5$.

$$x = -5 \quad \text{or} \quad x = 5$$ • Solve the absolute value equation.

The solutions are -5 and 5.

We will refer to the preceding method of solving a quadratic equation as the **square root procedure.**

The Square Root Procedure

If $x^2 = c$, then $x = \sqrt{c}$ or $x = -\sqrt{c}$, which can also be written as $x = \pm\sqrt{c}$.

EXAMPLE

If $x^2 = 9$, then $x = \sqrt{9} = 3$ or $x = -\sqrt{9} = -3$. This can be written as $x = \pm 3$.

If $x^2 = 7$, then $x = \sqrt{7}$ or $x = -\sqrt{7}$. This can be written as $x = \pm\sqrt{7}$.

If $x^2 = -4$, then $x = \sqrt{-4} = 2i$ or $x = -\sqrt{-4} = -2i$. This can be written as $x = \pm 2i$.

INTERVENTION FOR CALCULUS

EXAMPLE 2 Solve by Using the Square Root Procedure

Use the square root procedure to solve each equation.

a. $3x^2 + 12 = 0$ **b.** $(x + 1)^2 = 48$

Solution

a. $3x^2 + 12 = 0$

$3x^2 = -12$ • Solve for x^2.

$x^2 = -4$

$x = \pm\sqrt{-4}$ • Take the square root of each side of the equation and insert a plus-or-minus sign in front of the radical.

$x = -2i$ or $x = 2i$

The solutions are $-2i$ and $2i$.

b. $(x + 1)^2 = 48$

$x + 1 = \pm\sqrt{48}$ • Take the square root of each side of the equation and insert a plus-or-minus sign in front of the radical.

$x + 1 = \pm 4\sqrt{3}$ • Simplify.

$x = -1 \pm 4\sqrt{3}$

$x = -1 + 4\sqrt{3}$ or $x = -1 - 4\sqrt{3}$

The solutions are $-1 + 4\sqrt{3}$ and $-1 - 4\sqrt{3}$.

INTERVENTION FOR CALCULUS

ASSIGNMENT TASK 4

Solve by using the square root procedure

■**28.** $(x + 2)^2 + 28 = 0$

INTERVENTION FOR CALCULUS

Solving Quadratic Equations by Completing the Square

Consider two binomial squares and their perfect-square trinomial products.

Square of a Binomial	=	Perfect-Square Trinomial
$(x+5)^2$	=	$x^2+10x+25$
$(x-3)^2$	=	x^2-6x+9

In each of the preceding perfect-square trinomials, the coefficient of x^2 is 1 and the constant term is the square of half the coefficient of the x term.

$$x^2+10x+25,\quad \left(\frac{1}{2}\cdot 10\right)^2=25$$

$$x^2-6x+9,\quad \left(\frac{1}{2}\cdot(-6)\right)^2=9$$

Adding to a binomial of the form x^2+bx the constant term that makes the binomial a perfect-square trinomial is called **completing the square.** For example, to complete the square of x^2+8x, add

$$\left(\frac{1}{2}\cdot 8\right)^2=16$$

to produce the perfect-square trinomial $x^2+8x+16$.

Completing the square is a powerful procedure that can be used to solve *any* quadratic equation. For instance, to solve $x^2-6x+13=0$, first isolate the variable terms on one side of the equation and the constant term on the other side.

$x^2-6x=-13$ • Subtract 13 from each side of the equation.

$x^2-6x+9=-13+9$ • Complete the square by adding $\left[\frac{1}{2}(-6)\right]^2=9$ to each side of the equation.

$(x-3)^2=-4$ • Factor and solve by the square root procedure.

$x-3=\pm\sqrt{-4}$

$x-3=\pm 2i$

$x=3\pm 2i$

The solutions of $x^2-6x+13=0$ are $3-2i$ and $3+2i$. You can check these solutions by substituting each solution into the original equation. For instance, the following check shows that $3-2i$ does satisfy the original equation.

$x^2-6x+13=0$

$(3-2i)^2-6(3-2i)+13\overset{?}{=}0$ • Substitute $3-2i$ for x.

$9-12i+4i^2-18+12i+13\overset{?}{=}0$ • Simplify.

$4+4(-1)\overset{?}{=}0$

$0=0$ • The left side equals the right side, so $3-2i$ checks.

EXAMPLE 3 Solve by Completing the Square

Solve $x^2 = 2x + 6$ by completing the square.

Solution

$$x^2 = 2x + 6$$

$$x^2 - 2x = 6$$ • Isolate the constant term.

$$x^2 - 2x + 1 = 6 + 1$$ • Complete the square.

$$(x - 1)^2 = 7$$ • Factor and simplify.

$$x - 1 = \pm\sqrt{7}$$ • Apply the square root procedure.

$$x = 1 \pm \sqrt{7}$$ • Solve for x.

The exact solutions of $x^2 = 2x + 6$ are $1 - \sqrt{7}$ and $1 + \sqrt{7}$. A calculator can be used to show that $1 - \sqrt{7} \approx -1.646$ and $1 + \sqrt{7} \approx 3.646$. The decimals -1.646 and 3.646 are approximate solutions of $x^2 = 2x + 6$.

Completing the square by adding the square of half the coefficient of the x term requires that the coefficient of the x^2 term be 1. If the coefficient of the x^2 term is not 1, then first multiply each term on each side of the equation by the reciprocal of the coefficient of x^2 to produce a coefficient of 1 for the x^2 term.

EXAMPLE 4 Solve by Completing the Square

Solve $2x^2 + 8x - 1 = 0$ by completing the square.

Solution

$2x^2 + 8x - 1 = 0$

$2x^2 + 8x = 1$ • Isolate the constant term.

$\frac{1}{2}(2x^2 + 8x) = \frac{1}{2}(1)$ • Multiply both sides of the equation by the reciprocal of the coefficient of x^2.

$x^2 + 4x = \frac{1}{2}$

$x^2 + 4x + 4 = \frac{1}{2} + 4$ • Complete the square.

$(x + 2)^2 = \frac{9}{2}$ • Factor and simplify.

$x + 2 = \pm\sqrt{\frac{9}{2}}$ • Apply the square root procedure.

$x = -2 \pm 3\sqrt{\frac{1}{2}}$ • Solve for x.

$x = -2 \pm 3\frac{\sqrt{2}}{2}$ • Simplify.

$x = \frac{-4 \pm 3\sqrt{2}}{2}$

The solutions are $\frac{-4 + 3\sqrt{2}}{2}$ and $\frac{-4 - 3\sqrt{2}}{2}$.

INTERVENTION FOR CALCULUS

ASSIGNMENT TASK 5

Solve by completing the square

■ 42. $2x^2 + 10x - 3 = 0$

Solving Quadratic Equations by Using the Quadratic Formula

Completing the square for $ax^2 + bx + c = 0$ $(a \neq 0)$ produces a formula for x in terms of the coefficients a, b, and c. The formula is known as the *quadratic formula*, and it can be used to solve *any* quadratic equation.

The Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As a general rule, you should first try to solve quadratic equations by factoring. If the factoring process proves difficult, then solve by using the quadratic formula.

EXAMPLE 5 Solve by Using the Quadratic Formula

Use the quadratic formula to solve each of the following.

a. $x^2 = 3x + 5$ **b.** $4x^2 - 4x + 3 = 0$

Solution

a. $x^2 = 3x + 5$

$x^2 - 3x - 5 = 0$ • Write the equation in standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ • Use the quadratic formula.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$$ • $a = 1$, $b = -3$, $c = -5$.

$$= \frac{3 \pm \sqrt{29}}{2}$$

$$x = \frac{3 - \sqrt{29}}{2} \quad \text{or} \quad \frac{3 + \sqrt{29}}{2}$$

The solutions are $\frac{3 - \sqrt{29}}{2}$ and $\frac{3 + \sqrt{29}}{2}$.

b. $4x^2 - 4x + 3 = 0$

• The equation is in standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

• Use the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(3)}}{2(4)}$$

• $a = 4$, $b = -4$, $c = 3$.

$$= \frac{4 \pm \sqrt{16 - 48}}{2(4)} = \frac{4 \pm \sqrt{-32}}{8}$$

$$= \frac{4 \pm 4i\sqrt{2}}{8}$$

$$x = \frac{4 - 4i\sqrt{2}}{8} = \frac{1}{2} - \frac{\sqrt{2}}{2}i \quad \text{or} \quad x = \frac{4 + 4i\sqrt{2}}{8} = \frac{1}{2} + \frac{\sqrt{2}}{2}i$$

The solutions are $\frac{1}{2} - \frac{\sqrt{2}}{2}i$ and $\frac{1}{2} + \frac{\sqrt{2}}{2}i$.

INTERVENTION FOR CALCULUS

ASSIGNMENT TASK 6

Use quadratic formula to solve

■ 58. $2x^2 + 4x = 1$

INTERVENTION FOR CALCULUS

The Discriminant of a Quadratic Equation

The solutions of $ax^2 + bx + c = 0$, $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the radical, $b^2 - 4ac$, is called the **discriminant** of the equation $ax^2 + bx + c = 0$. If $b^2 - 4ac \geq 0$, then $\sqrt{b^2 - 4ac}$ is a real number. If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Thus the sign of the discriminant can be used to determine whether the solutions of a quadratic equation are real numbers.

The Discriminant and the Solutions of a Quadratic Equation

The equation $ax^2 + bx + c = 0$, with real coefficients and $a \neq 0$, has as its discriminant $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has *two distinct real solutions.*
- If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has *one real solution*. The solution is a double solution.
- If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has *two distinct nonreal complex solutions.* The solutions are conjugates of each other.

EXAMPLE 6 Use the Discriminant to Determine the Number of Real Solutions

For each equation, determine the discriminant and state the number of real solutions.

a. $2x^2 - 5x + 1 = 0$

b. $3x^2 + 6x + 7 = 0$

c. $x^2 + 6x + 9 = 0$

Solution

a. The discriminant of $2x^2 - 5x + 1 = 0$ is $b^2 - 4ac = (-5)^2 - 4(2)(1) = 17$. Because the discriminant is positive, $2x^2 - 5x + 1 = 0$ has two distinct real solutions.

b. The discriminant of $3x^2 + 6x + 7 = 0$ is $b^2 - 4ac = 6^2 - 4(3)(7) = -48$. Because the discriminant is negative, $3x^2 + 6x + 7 = 0$ has no real solutions.

c. The discriminant of $x^2 + 6x + 9 = 0$ is $b^2 - 4ac = 6^2 - 4(1)(9) = 0$. Because the discriminant is 0, $x^2 + 6x + 9 = 0$ has one real solution.

INTERVENTION FOR CALCULUS

Applications of Quadratic Equations

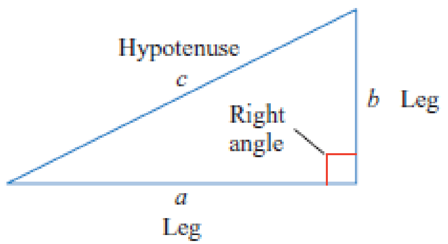
A **right triangle** contains one 90° angle. The side opposite the 90° angle is called the **hypotenuse.** The other two sides are called **legs.** The lengths of the sides of a right triangle are related by a theorem known as the Pythagorean Theorem.

The Pythagorean Theorem states that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs. This theorem is often used to solve applications that involve right triangles.

INTERVENTION FOR CALCULUS

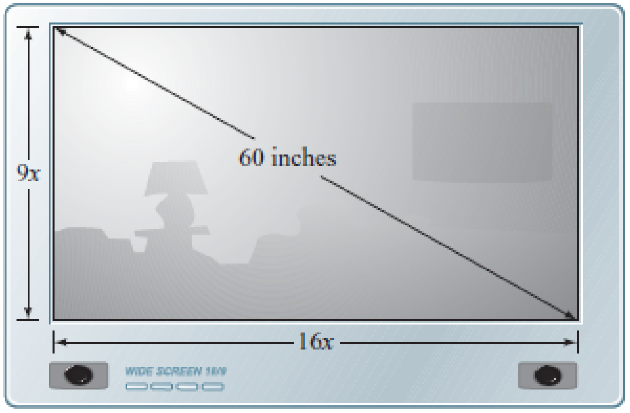
The Pythagorean Theorem

If a and b denote the lengths of the legs of a right triangle and c the length of the hypotenuse, then $c^2 = a^2 + b^2$.



EXAMPLE 7 Determine the Dimensions of a Television Screen

A television screen measures 60 inches diagonally, and its *aspect ratio* is 16 to 9. This means that the ratio of the width of the screen to the height of the screen is 16 to 9. Find the width and height of the screen.



A 60-inch television screen with a 16:9 aspect ratio.

Solution

Let $16x$ represent the width of the screen and let $9x$ represent the height of the screen. Applying the Pythagorean Theorem gives us

$$\begin{aligned}(16x)^2 + (9x)^2 &= 60^2 \\ 256x^2 + 81x^2 &= 3600 \\ 337x^2 &= 3600 \\ x^2 &= \frac{3600}{337} \\ x &= \sqrt{\frac{3600}{337}} \approx 3.268 \text{ inches}\end{aligned}$$

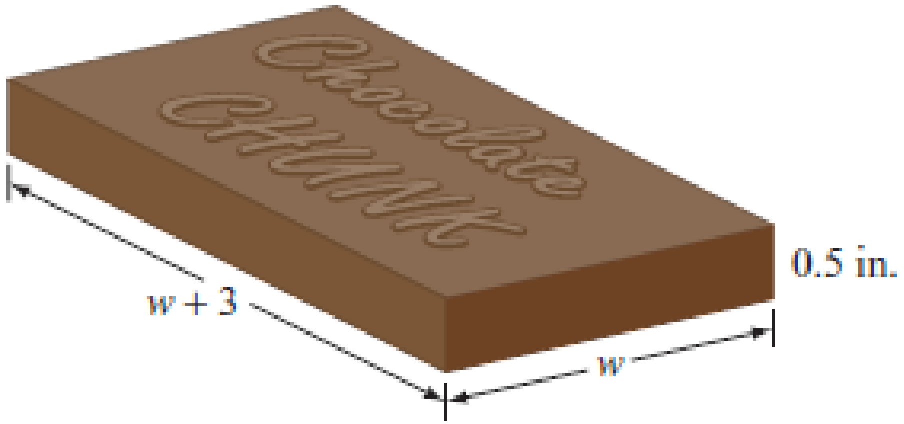
- Solve for x.
- Apply the square root procedure. The plus-or-minus sign is not used in this application because we know x is positive.

The height of the screen is about $9(3.268) \approx 29.4$ inches, and the width of the screen is about $16(3.268) \approx 52.3$ inches.

INTERVENTION FOR CALCULUS

EXAMPLE 8 Determine the Dimensions of a Candy Bar

A company makes rectangular solid candy bars that measure 5 inches by 2 inches by 0.5 inch. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should the dimensions be for the new candy bar if the company keeps the height at 0.5 inch and makes the length of the candy bar 3 inches longer than the width?



Solution

The volume of a rectangular solid is given by $V = lwh$. The original candy bar had a volume of $5 \cdot 2 \cdot 0.5 = 5$ cubic inches. The new candy bar will have a volume of $80\%(5) = 0.80(5) = 4$ cubic inches.

Let w represent the width and $w + 3$ represent the length of the new candy bar. For the new candy bar we have

$$lwh = V$$

$$(w + 3)(w)(0.5) = 4$$ • Substitute in the volume formula.

$$(w + 3)(w) = 8$$ • Multiply each side by 2.

$$w^2 + 3w = 8$$ • Simplify the left side.

$$w^2 + 3w - 8 = 0$$ • Write in $ax^2 + bx + c = 0$ form.

$$w = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-8)}}{2(1)}$$ • Use the quadratic formula.

$$= \frac{-3 \pm \sqrt{41}}{2}$$

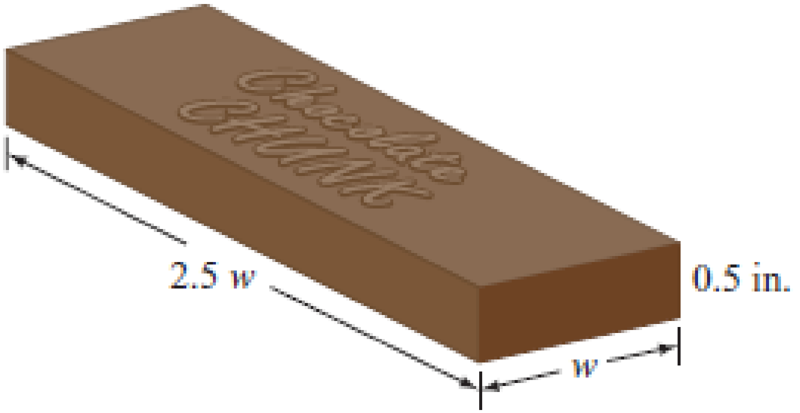
$$\approx 1.7 \quad \text{or} \quad -4.7$$

We can disregard the negative value because the width must be positive. The width of the new candy bar, to the nearest tenth of an inch, should be 1.7 inches. The length should be 3 inches longer, which is 4.7 inches.

INTERVENTION FOR CALCULUS

ASSIGNMENT TASK 7 Solve the problem

■94. **Dimensions of a Candy Bar** A company makes rectangular solid candy bars that measure 5 inches by 2 inches by 0.5 inch. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should be the dimensions, to the nearest tenth of an inch, of the new candy bar if the company decides to keep the height at 0.5 inch and to make the length of the new candy bar 2.5 times as long as its width?



INTERVENTION FOR CALCULUS

Equations That Are Quadratic in Form

The equation $4x^4 - 25x^2 + 36 = 0$ is said to be **quadratic in form,** which means that it can be written in the form

$$au^2 + bu + c = 0, \quad a \neq 0$$

where u is an algebraic expression involving x. For example, if we make the substitution $u = x^2$ (which implies that $u^2 = x^4$), then our original equation can be written as

$$4u^2 - 25u + 36 = 0$$

This quadratic equation can be solved for u, and then, using the relationship $u = x^2$, we can find the solutions of the original equation.

EXAMPLE 6 Solve an Equation That Is Quadratic in Form

Solve: $4x^4 - 25x^2 + 36 = 0$

Solution

Make the substitutions $u = x^2$ and $u^2 = x^4$ to produce the quadratic equation $4u^2 - 25u + 36 = 0$. Factor the quadratic polynomial on the left side of the equation.

$$(4u - 9)(u - 4) = 0$$

$$4u - 9 = 0 \quad \text{or} \quad u - 4 = 0$$

$$u = \frac{9}{4} \qquad u = 4$$

Substitute x^2 for u to produce

$$x^2 = \frac{9}{4} \quad \text{or} \quad x^2 = 4$$

$$x = \pm\sqrt{\frac{9}{4}} \qquad x = \pm\sqrt{4}$$

$$x = \pm\frac{3}{2} \qquad x = \pm 2$$

• Check in the original equation.

The solutions are $-2, -\frac{3}{2}, \frac{3}{2}$, and 2.

The following table shows equations that are quadratic in form. Each equation is accompanied by an appropriate substitution that will enable it to be written in the form $au^2 + bu + c = 0$.

Equations That Are Quadratic in Form

Original Equation	Substitution	$au^2 + bu + c = 0$ Form
$x^4 - 8x^2 + 15 = 0$	$u = x^2$	$u^2 - 8u + 15 = 0$
$x^6 + x^3 - 12 = 0$	$u = x^3$	$u^2 + u - 12 = 0$
$x^{1/2} - 9x^{1/4} + 20 = 0$	$u = x^{1/4}$	$u^2 - 9u + 20 = 0$
$2x^{2/3} + 7x^{1/3} - 4 = 0$	$u = x^{1/3}$	$2u^2 + 7u - 4 = 0$
$15x^{-2} + 7x^{-1} - 2 = 0$	$u = x^{-1}$	$15u^2 + 7u - 2 = 0$

EXAMPLE 9 Solve a Work Problem

A small pipe takes 12 minutes longer than a larger pipe to empty a tank. Working together, they can empty the tank in 1.75 minutes. How long would it take the smaller pipe to empty the tank if the larger pipe is closed?

Solution

Let t be the time it takes the smaller pipe to empty the tank. Then $t - 12$ is the time for the larger pipe to empty the tank. Both pipes are open for 1.75 minutes. Therefore, $\frac{1.75}{t}$ is the portion of the tank emptied by the smaller pipe and $\frac{1.75}{t-12}$ is the portion of the tank emptied by the larger pipe. Working together, they empty one tank. Thus $\frac{1.75}{t} + \frac{1.75}{t-12} = 1$. Solve this equation for t.

$$\frac{1.75}{t} + \frac{1.75}{t-12} = 1$$

$$t(t-12)\left(\frac{1.75}{t} + \frac{1.75}{t-12}\right) = t(t-12)\cdot 1$$

• Multiply each side by the LCD $t(t-12)$.

$$1.75(t-12) + 1.75t = t^2 - 12t$$

$$1.75t - 21 + 1.75t = t^2 - 12t$$

$$0 = t^2 - 15.5t + 21$$

• Write the quadratic equation in standard form.

Using the quadratic formula, the solutions of the above equation are $t = 1.5$ and $t = 14$. Substituting $t = 1.5$ into the time for the larger pipe would give a negative time $(1.5 - 12 = -10.5)$, so that answer is not possible. The time for the smaller pipe to empty the tank with the larger pipe closed is 14 minutes.

REFERENCES:

- Larson, Ron and Falvo, David C. Algebra and Trigonometry
 Eight Edition,
- Blitzer, Robert , Precalculus , 5th edition